2749

Fluid Dynamics =
flows of liquids and gases.

biological examples

- swimming and flying
- blood flow
- Cytoskeleton fluid-like on long time-scales.

physics

- fluid
  - condensed matter.
  - translational + rotational invariant
  - coarse grained theory involving small number of effective degrees of freedom.

forces $\longleftrightarrow$ flows

Assumption	Counter-example
• iso-thermal	convective flow
• single component	vinaigrette
• locally structureless	liquid crystal.

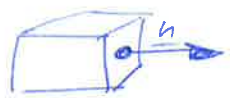
$\rho(\underline{x}) \equiv$ local fluid density

$\underline{v}(\underline{x}) \equiv$ flow field

$\underline{\underline{\sigma}}(x) \equiv$ stress tensor $[N/m^2]$

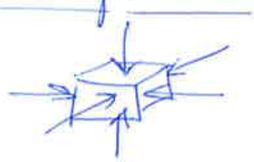
- rank 2 (3×3)
- contact forces acting on fluid element

fluid element



$\underline{\underline{\sigma}} \cdot \underline{n} \equiv$ force density acting on boundary

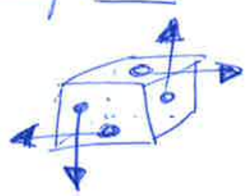
example 1:



pressure

$$\sigma = -p\,\mathbb{I} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

example 2



shear ~~force~~ stress

$$\sigma = \begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notation: Landau-Lifshitz + Happel-Brenner

Why stress tensor symmetric?

- stress tensor = contact force
- anti-symmetric part
  = external torque 
  ≠ contact force.

- external torque on fluid element would result in rigid body rotation, irrelevant for coarse-grained description of unstructured fluid.

- for structured fluids (e.g. liquid crystals), external angular momentum can be converted into internal angular momentum

The strain rate tensor characterizes "deformations" of fluid elements.

$$2\underline{\underline{\Gamma}} = \underline{\nabla}\,\underline{v} + (\underline{\nabla}\,\underline{v})^T$$

$$\Gamma_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$$

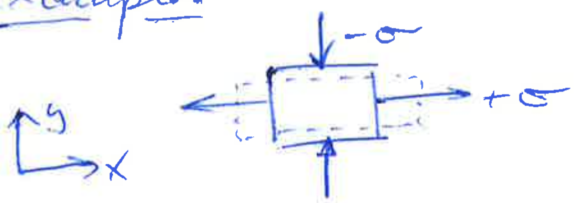
- pure translation $\Gamma = 0$
- pure rotation $\Gamma = 0$.
- dilation part: $\Gamma_0 = \frac{1}{3}\,\mathrm{tr}\,\Gamma \cdot \underline{\mathbb{I}}$

  $\mathrm{tr}\,\Gamma \equiv$ rate of volume change of fluid element
- shear

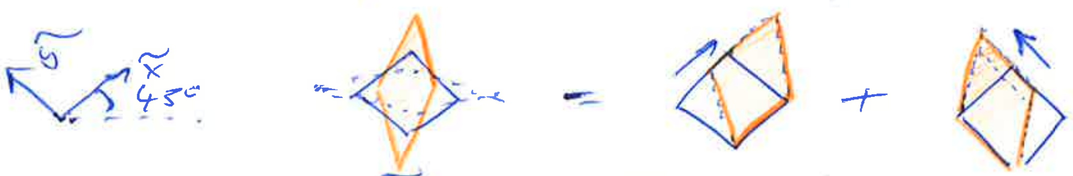
$$\Delta = \Gamma - \frac{1}{3}\,\mathrm{tr}\,\Gamma \cdot \underline{\mathbb{I}}$$

$\equiv$ shear rate tensor

<u>examples</u>



$$\Delta = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\tilde{\Delta} = R_z(45°)\,\Delta\, R_z(45°) = \begin{pmatrix} 0 & -\sigma & 0 \\ -\sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Pi = -\sigma + \rho \underline{v} \otimes \underline{v}$$

$\equiv$ "momentum flux tensor" (LL)

$\equiv$ "total stress tensor" [HB]

Structure of stress tensor σ under reversal of time arrow.

$t \longrightarrow -t$

$\underline{v} \longrightarrow -\underline{v}$

$p \longrightarrow p$

$\sigma' \longrightarrow -\sigma'$

$$\sigma = \underbrace{-p\,\mathbb{I}}_{\substack{\text{reversible} \\ \text{part} \equiv \\ \text{hydrostat.} \\ \text{pressure}}} + \underbrace{\sigma'}_{\substack{\text{irreversible part} \\ \equiv \text{viscous stress}}}$$

couples to "deformations" of fluid elements, resulting in dissipation

How to couple
- stress &
- strain rate tensor ?

σ

Constitutive eqn. (dep. on material properties)

Mass conservation
Newtons law

Γ

Mass conservation

$\rho \underline{v} \equiv$ mass flux.

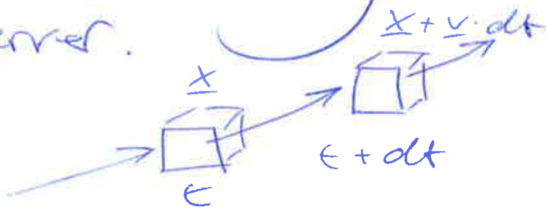
Partial time derivative. $\frac{\partial}{\partial t}$

... measured by observer fixed in space.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v}) = -\underbrace{\underline{v} \cdot \nabla \rho}_{\text{convection}} - \underbrace{\rho\, \underline{\nabla} \cdot \underline{v}}_{\text{velocity divergence}}.$$

Substantial time derivative $\frac{D}{Dt}$

... measured by co-flowing observer.



$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla.$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho \underbrace{\underline{\nabla} \cdot \underline{v}}_{\text{sources/ sinks of velocity field.}}$$

Momentum conservation (Newton's law).

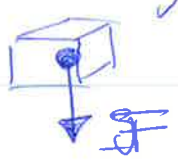
$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \mathbf{f}$$

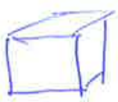
mass-per-unit-volume times acceleration

contact forces

body forces (force volume density)

e.g. $\mathbf{f} = \rho \cdot \mathbf{g}$





$$\oint \sigma \cdot d\mathbf{S} = \int \nabla \cdot \sigma \, dV$$

Sum of contact forces $\longrightarrow$ volume integral

Mass conservation + momentum conservation

$$\frac{\partial(\rho \underline{v})}{\partial t} = \frac{D(\rho \underline{v})}{Dt} - \underline{v}\cdot(\underline{\nabla}\rho\underline{v})$$

$$= \frac{D\rho}{Dt}\underline{v} + \rho\frac{D\underline{v}}{Dt} - \underline{v}\cdot(\nabla\rho\underline{v})$$

$$= -\rho(\underline{\nabla}\cdot\underline{v})\underline{v} + \underline{\nabla}\cdot\underline{\underline{\sigma}} - \underline{v}\cdot(\nabla\rho\underline{v})$$

$$= -\nabla(\rho\underline{v}\otimes\underline{v}) + \nabla\cdot\sigma$$

$$= -\nabla\overline{\pi}\,, \quad \overline{\pi} = \rho\underline{v}\otimes\underline{v} - \sigma$$

$$\boxed{\frac{\partial(\rho\underline{v})}{\partial t} = -\nabla\overline{\pi}}$$ Euler eqn.

Viscous stresses coupled to "deformations" of fluid elements result in the production of entropy

$s \equiv$ entropy per unit mass

$$\rho T \frac{Ds}{Dt} = \underbrace{\underline{\underline{\Gamma}} : \underline{\underline{\sigma}}'}_{\text{internal dissipation}} + \underbrace{k \nabla^2 T}_{\text{thermal conduction}}$$

(LL §49)

$\underline{\underline{\sigma}} \cdot \underline{n} \equiv$ contact forces

$\oint \underline{v} \cdot \underline{\underline{\sigma}} \cdot \underline{n} = \int \underline{\nabla} \cdot (\underline{v} \cdot \underline{\underline{\sigma}}) \equiv$ work done by contact forces (per unit volume & unit time)

$$\underline{\nabla} \cdot (\underline{v} \cdot \underline{\underline{\sigma}}) = \underbrace{-(\underline{\nabla} \cdot \underline{v}) \cdot p}_{\text{volume work}} + \underbrace{(\underline{\nabla}\, \underline{v}) : \underline{\underline{\sigma}}'}_{\substack{\Gamma : \sigma' \\ \text{internal dissipation}}} + \underbrace{\underline{v} \cdot (\underline{\nabla} \cdot \underline{\underline{\sigma}})}_{\text{work done in moving volume element as a whole}}$$

Onsager pair of "conj. variables"

$$\underset{\text{flux}}{\underline{\underline{\Gamma}}} \longleftrightarrow \underset{\text{force}}{\underline{\underline{\sigma}}'}$$

Exploiting tensor symmetries

$$\sigma' = \frac{1}{3}\operatorname{tr}\sigma'\,\mathbb{I} + \underbrace{\left(\sigma' - \frac{1}{3}\operatorname{tr}\sigma'\,\mathbb{I}\right)}_{\bar{\sigma}'}$$

$$\Gamma = \frac{1}{3}\underbrace{\operatorname{tr}\Gamma}_{\underline{\nabla}\cdot\underline{v}}\,\mathbb{I} + \underbrace{\left(\Gamma - \frac{1}{3}\operatorname{tr}\Gamma\,\mathbb{I}\right)}_{\Delta}$$

$\hat{=}$ shear rate tensor

$$\sigma' : \Gamma = \underbrace{\frac{1}{3}\operatorname{tr}\sigma'\,\mathbb{I} : \frac{1}{3}\operatorname{tr}\Gamma\,\mathbb{I}}_{\frac{1}{3}\operatorname{tr}\sigma'\,(\underline{\nabla}\cdot\underline{v})}$$

energy dissipation due to volume change

$$+ \underbrace{\bar{\sigma}' : \Delta}$$

energy dissipation due to shear deformations.

Constitutive equation

$$\sigma' = \sigma'(\Gamma)$$

- ideal fluid: $\sigma' = 0$.

- Newtonian fluid.

$$\sigma' = \text{lin. fct. of } \Gamma$$

$$\sigma'_{ij} = C_{ijke}\,\Gamma_{ke}$$

$$= \underbrace{k}_{\text{volume viscosity}}(\underline{\nabla}\cdot\underline{v})\,\underline{\underline{I}} + 2\underbrace{\eta}_{\text{shear viscosity}}\,\Delta$$

Incompressible fluids $k \to \infty$: $\underline{\nabla}\cdot\underline{v} = 0$.

- Non-Newtonian fluids.
  - shear softening   Ketchup
  - shear stiffening   polymer melts
  - memory

- water $\eta = 1.0\ \frac{pN\cdot ms}{\mu m^2}$ @ 24°C

  $0.7\ \frac{pN\cdot ms}{\mu m^2}$ @ 36°C.

Navier-Stokes equation for incompressible Newtonian fluid.

- Euler eqn.

$$\frac{\partial(\rho \underline{v})}{\partial t} = -\underline{\nabla} \cdot \underline{\underline{\Pi}}$$

$$\underline{\underline{\Pi}} = -\underline{\underline{\sigma}} + \rho\, \underline{v} \otimes \underline{v}$$

$$\underline{\underline{\sigma}} = -p\, \underline{\underline{I}} + \underline{\underline{\sigma}}'$$

- Incompressible $\underline{\nabla} \cdot \underline{v} = 0$

$\rho(\underline{r}) = \text{const.}$

- Newtonian + incomp.

$$\sigma' = 2\eta \underline{\underline{\Delta}} = 2\eta \Gamma$$

$$\rho \frac{\partial \underline{v}}{\partial t} = -\rho \underbrace{\underline{\nabla} \cdot (\underline{v} \otimes \underline{v})}_{\underbrace{(\underline{\nabla} \cdot \underline{v})}_{=0} \underline{v} + \underline{v} \cdot \nabla \underline{v}} - \nabla p + 2\eta \underbrace{\underline{\nabla}\, \Gamma}_{\underline{\nabla} \frac{1}{2}[\underline{\nabla}\, \underline{v} + (\underline{\nabla}\, \underline{v})^T]}$$

$$= \frac{1}{2}[\underline{\nabla}^2 \underline{v} + \underline{\nabla}\, \underbrace{(\underline{\nabla} \cdot \underline{v})}_{=0}]$$

$$\rho \frac{D \underline{v}}{Dt} = \rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right) = -\nabla p + \eta \underline{\nabla}^2 \underline{v}.$$

Hydrodynamic dissipation for incompressible fluids.

work done on fluid volume → compression

work done on fluid volume → heat

$\underline{v}\cdot\underline{\underline{\sigma}}\cdot d\underline{S}$

$\equiv$ work done by contact force on surface element

* $$\oint \underline{v}\cdot\underline{\underline{\sigma}}\cdot d\underline{S} = \int \nabla\cdot(\underline{v}\cdot\underline{\underline{\sigma}})\,d^3\underline{r}$$

* $$\sigma = -p\,\mathbb{I} + 2\eta\,\underline{\underline{\Gamma}}$$

* $$\nabla\cdot\sigma = -\nabla p\,\mathbb{I} + \nabla\cdot\eta\left[\nabla\underline{v} + (\nabla\underline{v})^T\right]$$

$$= \underbrace{-\nabla p\,\mathbb{I} + \eta\nabla^2\underline{v}}_{=0} + \nabla\underbrace{(\nabla\cdot\underline{v})}_{=0}$$

$$\nabla\cdot(\underline{v}\cdot\underline{\underline{\sigma}}) = (\nabla\underline{v}):\sigma + \underbrace{\underline{v}\cdot(\nabla\cdot\underline{\underline{\sigma}})}_{=0}$$

$$= \nabla\underline{v}:(-p\,\mathbb{I} + 2\eta\,\underline{\underline{\Gamma}})$$

$$= -p\nabla\cdot\underline{v} + 2\eta\nabla\underline{v}:\underline{\underline{\Gamma}}$$

$$= 2\eta\,\underline{\underline{\Gamma}}:\underline{\underline{\Gamma}}$$

Note: For compressible fluids, extra terms

Reynolds number

$$\rho \frac{D\underline{v}}{Dt} = -\nabla p + \mu \nabla^2 \underline{v} + \underline{F}.$$

L ... length-scale
V_0 .... velocity scale

$\sim \rho \cdot V_0^2/L$ — inertial forces
$\sim \mu V_0/L^2$ — viscous forces

$$Re = \frac{\text{inertial forces}}{\text{viscous forces}} = \frac{\rho V_0^2/L}{\eta V_0/L^2}$$

$$= \frac{\rho V_0 L}{\eta}$$

examples:

Sperm in water

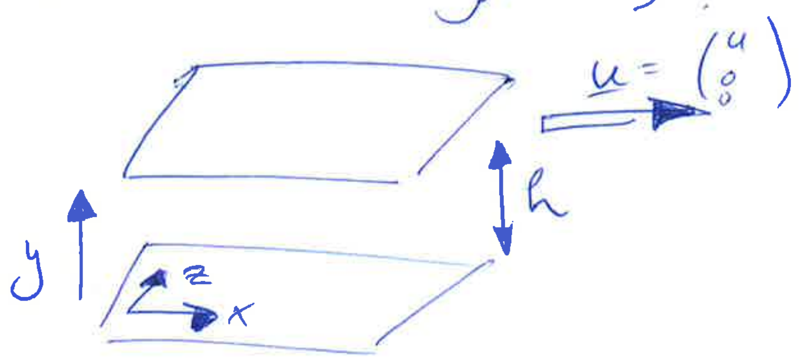
$\rho = 1\,kg/l$, $\eta = 10^{-3}\,Pa\cdot s = 1 \frac{pN}{\mu m^2}\cdot ms$

$V_0 = 100\,\mu m/s$, $L = 100\,\mu m$

$\longrightarrow$

$Re = \underline{\underline{10^{-2}}}$

Scherströmung zwischen zwei Platten (LL §17)



Re = 0: $$0 = -\nabla p + \eta \Delta v. \qquad (*)$$

Symmetrie: $$p = p(y), \quad \underline{v} = \begin{pmatrix} v_x(y) \\ 0 \\ 0 \end{pmatrix}$$

$$0 = \begin{pmatrix} 0 \\ \partial_y p \\ 0 \end{pmatrix} + \eta \begin{pmatrix} \partial_y^2 v_x \\ 0 \\ 0 \end{pmatrix} \qquad (*')$$

$$\partial_y p = 0 \Rightarrow p = \text{const.}$$

$$v_x = ay + b. = \frac{y}{h} u.$$

$$\nabla v = \begin{pmatrix} 0 & u/h & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma' = \eta [\nabla v + (\nabla v)^T] = \begin{pmatrix} 0 & u/h & 0 \\ u/h & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\sigma = -p \mathbb{I} + \sigma'$$

$$\sigma_{yy} = p$$

$$\sigma_{xy} = \frac{\eta u}{h}$$

Reminder:

In-compressible
Newtonian fluid.

$$\mathrm{tr}\,\Gamma = \nabla \cdot v = 0$$

$$\mathrm{tr}\,\sigma' = k \cdot \mathrm{tr}\,\Gamma = 0$$

$$\sigma' = \sigma' - \frac{1}{3}\mathrm{tr}\,\sigma' I = 2\eta\,\Delta = 2\eta\left(\Gamma - \frac{1}{3}\mathrm{tr}\,\Gamma\right)$$

$$Re = \frac{\rho V_0 L}{\eta}.$$

§17 Strömung durch ein Rohr.

$2R$

$$V = \begin{pmatrix} V_x(r) \\ 0 \\ 0 \end{pmatrix}$$

$$\partial_y p = \partial_z p = 0 \Rightarrow p = p(x)$$

$$\underbrace{\frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}}_{\text{dep. only on } y,z} = \underbrace{\frac{1}{\eta}\frac{dp}{dx}}_{\text{dep. only on } x}.$$

$$\Rightarrow \frac{dp}{dx} = \text{const.} = -\frac{\Delta p}{\ell}$$

Laplace operator in polar coords.

$$-\frac{\Delta p}{\eta \ell} = \Delta V = \frac{1}{r}\partial_r (r \partial_r V_x)$$

$$-\frac{\Delta p}{2\eta\ell} r^2 + a = r \partial_r V_x \qquad \text{simple integration}$$

$$-\frac{\Delta p}{4\eta\ell} r^2 + \underbrace{a \ln r}_{\text{singularity at } r=0} + b = V_x.$$

$$\Rightarrow V_x = \frac{\Delta p}{4\eta\ell}(R^2 - r^2)$$

flux

$$Q = 2\pi\rho \int_0^R r V_x \, dr$$

$$= \frac{\pi\rho\Delta p}{8\eta\ell} R^4$$

?

Poiseuillesches Gesetz

Tricks to solve flow problems at low Re

- potential flow = irrotational flow with $\nabla \times \underline{v} = 0$

$\Rightarrow \underline{v} = \nabla \Phi$.

$\Rightarrow$ solve equation for Φ (harmonic functions)

- axisymmetric flow

$\psi = \frac{Q}{2\pi}$

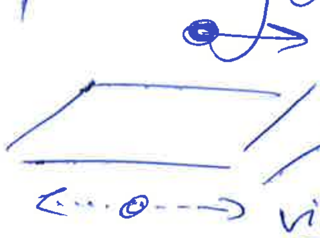
$\Rightarrow$ equation for ψ.

- 2D flow problems: formulate in terms of single complex coordinate.

  - conformal mappings

$z \mapsto z^n$.

- fundamental singularities
- methods of images



particle

no-slip boundary

virtual particle to cancel flow @ boundary

Stokes equation

$$0 = -\nabla p + \eta \nabla^2 \underline{v} + \underline{F}.$$

- linear PDE.
- fundamental solution = Oseen tensor

$$G_{ij} = \frac{1}{8\pi\eta} \frac{1}{r} \left(\delta_{ij} + \hat{r}_i \otimes \hat{r}_j \right)$$

$\equiv$ flow due to a point force

$\equiv$ Stokeslet.

$$\underline{v} = \underline{G} \cdot \underline{F}$$

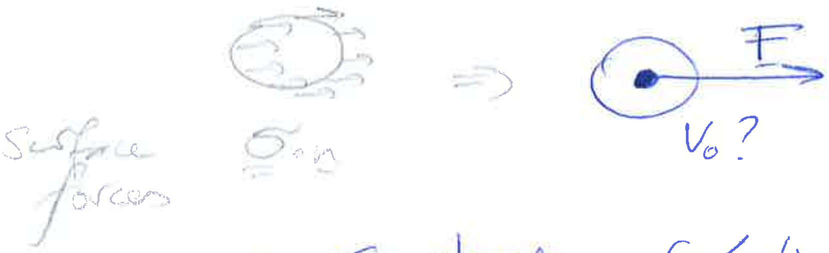
$$p = \frac{1}{4\pi} \frac{\underline{r} \cdot \underline{F}}{|r|^3}$$

- spatial derivatives of Oseen tensor also solutions of Stokes equation

dipole 

quadrupole ....

Drag on a sphere



- Construct solution from superposition of fundamental solutions

- flow axisymmetric & mirror symmetric wrt yz-plane

- Stokeslet $= G_{ij} F$

  Stokes doublet (dipole)

  not mirror-sym

- Stokes quadrupole

flow field around a ~~moving~~ translating sphere

$$\underline{V}_i = \left(1 + \frac{a^2}{6}\nabla^2\right) G_{ij}\, \underline{F}_j$$

$$p = \frac{1}{4\pi} \frac{\underline{r}\cdot\underline{F}}{r^3}$$ ≡ decays quadratically

stress tensor

$$\sigma = -p\,\underline{I} + 2\eta\,\Gamma$$

traction force

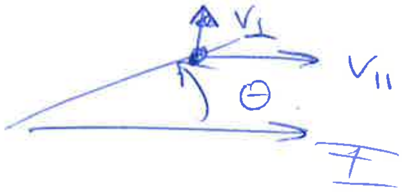
$$\sigma\cdot\underline{n} = -\frac{1}{4\pi a^2}\,\underline{F}\,.$$

surface force density exerted by fluid

[individual contributions $-p\underline{I}\cdot\underline{n}$ and $2\eta\Gamma\cdot\underline{n}$ more complicated]

Oseen tensor in spherical coords.

$v_i = G_{ij} \cdot F_j$



Stokeslet = axisymmetric flow

$$v_\parallel^{(1)} = \frac{3+\cos 2\theta}{16\pi\eta r} f_0$$

$$v_\perp^{(1)} = \frac{\sin 2\theta}{16\pi\eta r} f_0 .$$

Laplacian of Oseen tensor

$$v_\parallel^{(3)} = -\frac{1+3\cos 2\theta}{8\pi\eta r^3} \alpha f_0$$

$$v_\perp^{(3)} = -\frac{3\sin 2\theta}{8\pi\eta r^3} \alpha f_0 .$$

$$\alpha = \frac{a^2}{6} .$$

at $r=a$: $v_0 \overset{!}{=} v_\parallel = v_\parallel^{(1)} - \alpha v_\parallel^{(3)} \equiv \frac{1}{6\pi\eta a} f_0$

$0 \overset{!}{=} v_\perp = v_\perp^{(1)} - \alpha v_\perp^{(3)} \equiv 0 .$

$$\boxed{F = 6\pi\eta a v_0}$$

Multipole gymnastiques

reminder: electrostatic: charge distribution

dominates far-field ← - net charge ⊕
- dipole ⊕ ⊖
- quadrupole ....

We are dealing with force fields.

$$f_i(\underline{r}) = \sigma_{ij} \cdot n_j$$

surface density of forces

Monopol = net force

$$F_{tot,i} = \int_S d^2\underline{r}\; f_i(\underline{r})$$

Cartesian multipole moments

$$f_{j,\partial} = \int_S d^2\underline{r}\; (\underline{r}-\underline{r}_0)^{\partial} f_j(\underline{r})$$

Multipole expansion

$$f_j(\underline{r}) = \sum_{\partial} \frac{(-1)^{|\partial|}}{\partial!} f_{j,\partial} \nabla^{\partial} \delta(\underline{r}-\underline{r}_0)$$

Check:

$$f_{j,\partial} = \int_S d^2\underline{r}\,(\underline{r}-\underline{r}_0)^{\partial} \sum_k \frac{(-1)^k}{k!} f_{j,k} \nabla^k \delta(\underline{r}-$$

$$= f_{j,\partial}$$

Recover flow field:

$$V_i(\underline{r}) = \int d\underline{r}' \; G_{ij}(\underline{r}-\underline{r}')\, f_j(\underline{r}')$$

$$= \sum_{\partial} \frac{(-1)^{\partial}}{\partial!}\, G_{ij,\partial}(\underline{r}-\underline{r}_0)\, f_{j,\partial} .$$

Special case: sphere of radius a

$\longrightarrow$ Relation between multipoles, e.g.

$$\underbrace{f_{j,xx} + f_{j,yy} + f_{j,zz}}_{\text{one (out of several) trace of rank 3 tensor}} = a^2 f_j$$

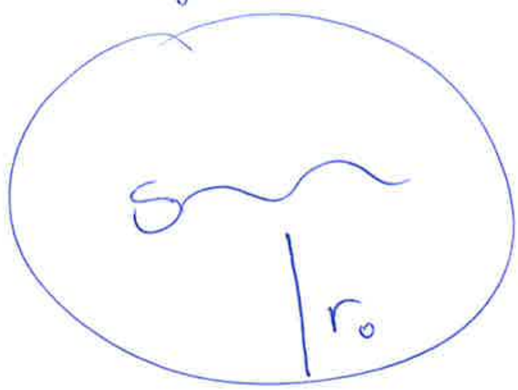
N.B.
These trace-relations are the only relations between multipoles (rank $k \longleftrightarrow$ rank $k+2$).
The "trace-less" tensors are independent and can be mapped on expansion into spherical harmonics.
Deep relation to representation theory of $SO(3)$

Application:
Flow field of a sphere

$$v_i = G_{ij}(\underline{r} - \underline{r}_0) F_j + \frac{a^2}{6} \nabla^2 G_{ij}(\underline{r} - \underline{r}_0) F_j + \ldots$$

In fact: "..." is zero, $\nabla^4 G_{ij} = 0$.

Thought experiment.

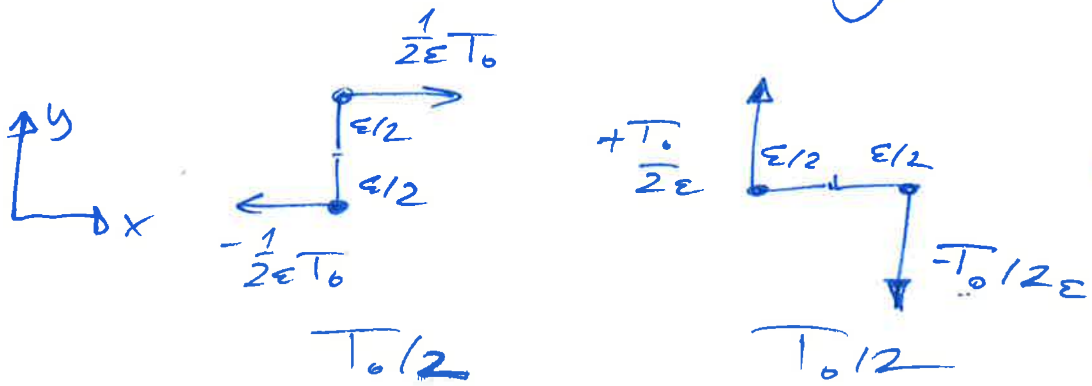


- remove swimmer + fluid sphere $|\underline{r}| \leq r_0$.
- add surface force density $\sigma(\underline{r}) \cdot \delta(|\underline{r}| - r_0)$

$\longrightarrow$ same flow field outside.

Rotation of a sphere

$\underline{T} = T_0 \, \underline{e}_z$ ≡ external angular momentum



force dipole representation

$$G_{ix,y}\, T_0/2 \; - \; G_{iy,x}\, T_0/2$$ flow field.

$$8\pi\eta \, G_{ix,y} = \begin{pmatrix} -y/r^3 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} x^2 \\ xy \\ xz \end{pmatrix} \frac{y}{r^5} + \begin{pmatrix} 0 \\ x/r^3 \\ 0 \end{pmatrix}$$

$$V_i = G_{ix,y}\, T_0/2 - G_{iy,x}\, T_0/2$$

$$= \frac{r_i}{r^3} \, \epsilon_{ijz} \, \frac{T_0}{8\pi\eta}$$

$$a\omega = \frac{T_0}{8\pi\eta a^2}$$

$$\boxed{8\pi\eta a^3 \cdot \omega = T_0}$$

Einstein relation

$$D_{trans} = \frac{k_B T}{6\pi\eta a} \quad \left[\frac{m^2}{s}\right]$$

$$D_{rot} = \frac{k_B T}{8\pi\eta a^3} \quad \left[\frac{1}{s}\right].$$

Example:

$a = 1\ \mu m$

$\eta = 1\ \frac{pN \cdot ms}{\mu m^2}$

$k_B T = 4 \cdot 10^{-21}\ J$

$= 4\ pN \cdot nm.$

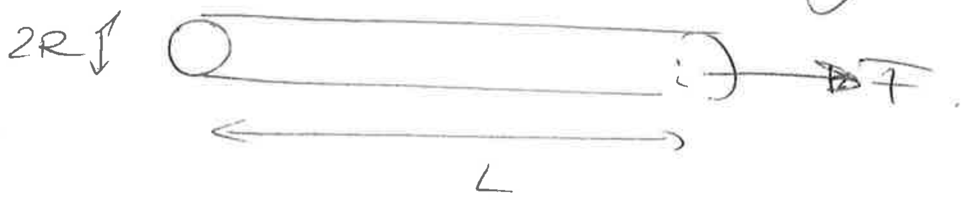
$D_{trans} = 0.2\ \frac{\mu m^2}{s}.$

$D_{rot} = 0.15\ \frac{1}{s}. \sim \frac{1}{6\,sec}.$

Swimming E. coli:

- transl. diffusion negligible
- rot. diffusion randomizes swimming direction within ca. 10 sec. $\longrightarrow$ adapted navigation

Drag on a translating cylinder



$N = \frac{L}{R}$.

$$f \approx \frac{F}{N}$$

$$f = 6\pi\eta R\,[v_{||} - v_{fluid}].$$

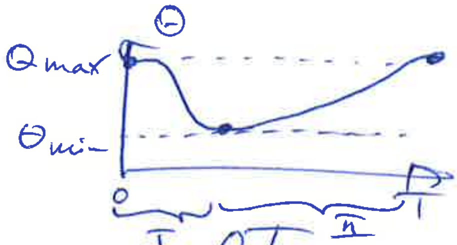
$$v_{fluid} = \sum_{\substack{k=1\\ k\neq N/2}}^{N} \frac{f}{8\pi\eta R}\underbrace{\frac{2}{|x_k - x_{N/2}|}}_{\underbrace{2\int_0^{L/2-R} dx/R\,\frac{1}{|x-L/2|}}_{\underbrace{2\left[-\ln\left(L/2-x\right)\right]_0^{L/2-R}}_{2[-\ln R + \ln L/2] = 2\ln L/2R}}}$$

$$= \frac{2f}{8\pi\eta R}\ln L/2R \gg v_0 \text{ for } L \gg R.$$

$F_{||} = \xi_{||} V_{||}$, $\xi_{||} = \frac{4\pi\eta L}{2\ln L/2R}$, $F_\perp = \xi_\perp V_\perp$, $\xi_\perp = 2\xi_{||}$

Purcell's Scallop.

$$V_x = g(\Theta)\dot{\Theta}$$



$$\bar{V} = \int_0^T dt \; V_x(t) = \int_0^T dt \; g(\Theta)\,\dot{\Theta}$$

$$= \underbrace{\int_{\Theta_{min}}^{\Theta_{max}} d\Theta \; g(\Theta)}_{II} - \underbrace{\int_{\Theta_{min}}^{\Theta_{max}} d\Theta \; g(\Theta)}_{I}$$

$$= \underline{\underline{0}}.$$

You cannot swim with 1 (linear) degree of freedom.

N.B.

With 2 DOF, net propulsion becomes possible!

DOF#1 $\sim \varepsilon$

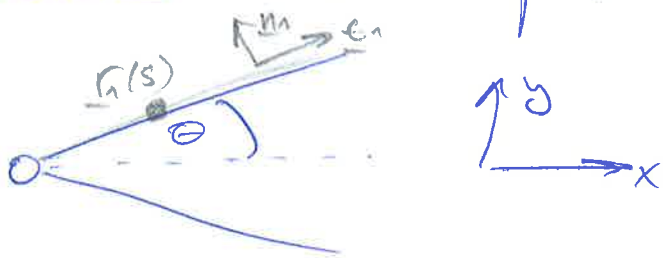
DOF#2 $\sim \varepsilon$

$\Rightarrow$ non. linear hydr. int. between DOFs

$\Rightarrow \bar{V} \sim \varepsilon^2$

Clams swims: two-steps-forward-one-step-back

Purcell's Scallop.



$$\underline{r}_1(s) = s\,\underline{t}_1\,, \quad \underline{t}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\underline{n}_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

- rigid body motion in x-direction

$$\underline{v} = v_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$f_1(s) = \xi_\parallel (\underline{v}\cdot\underline{t}_1)\underline{t}_1 + \xi_\perp (\underline{v}\cdot\underline{n}_1)\underline{n}_1$$

$$= \xi_\parallel v_0 \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \xi_\perp \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} (-\sin\theta) v_0$$

$$F_{tot} = \int_0^L ds\, f_1(s) + f_2(s)$$

$$= \begin{pmatrix} \xi_\parallel v_0 L \cdot 2\cos^2\theta + \xi_\perp v_0 L\, 2\sin^2\theta \\ 0 \end{pmatrix}$$

- forces exerted for shape change, motion of pivot point constrained

$$\underline{v}_1 = \dot{\theta} s\, \underline{n}_1$$

$$f_1(s) = \xi_\perp s \cdot \underline{n}_1 \cdot \dot{\theta}$$

$$F_{tot} = -2\xi_\perp L^2/2\, \dot{\theta} \sin\theta$$

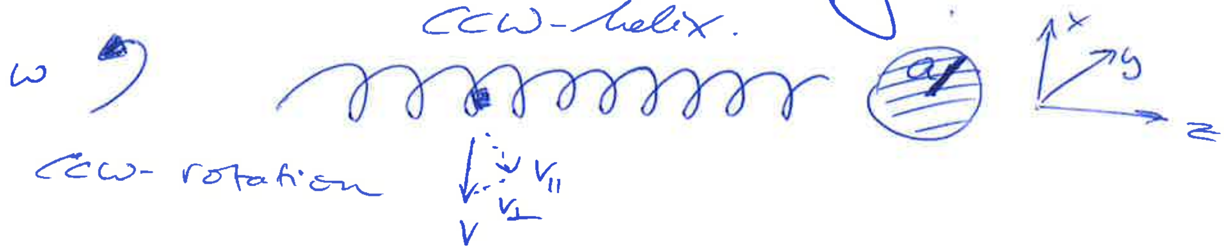
- force balance: $2\, v_0 L (\xi_\parallel \cos^2\theta + \xi_\perp \sin^2\theta)$

$$- \dot{\theta} L^2 \xi_\perp \sin\theta$$

$$\Rightarrow v_0 = +\frac{L}{2} \frac{\xi_\perp \sin\theta}{\xi_\parallel \cos^2\theta + \xi_\perp \sin^2\theta} \dot{\theta} = g(\theta)\dot{\theta}$$

Bacterial swimming

CCW-helix.



$$r(s) = \left(r_0 \cos\left(\frac{s}{\sqrt{r_0^2+\ell_0^2}} + \omega t\right), r_0 \sin\left(\frac{s}{\sqrt{r_0^2+\ell_0^2}} + \omega t\right), \ell_0 s + V_0 t \right)$$

$r_0 = \ell \cos\theta$, $\ell_0 = \ell \sin\theta$.

$\theta \equiv$ helix angle

- pure rotation for $V_0 = 0$

$\overline{F}_{\omega,z} = \int_0^L ds\, f(s) \cdot \underline{e}_z \equiv$ thrust force for constrained cell body

at $s=0$:

$\underline{t} = (0, \cos\theta, \sin\theta)$

$\underline{v} = (0, -r_0\omega, 0)$

$v_{\parallel} = (0, *, -r_0\omega \cos\theta \sin\theta)$

$\underline{v}_{\perp} = (0, *, r_0\omega \cos\theta \sin\theta)$.

$$f(s=0) = \zeta_{\parallel} v_{\parallel} + \zeta_{\perp} v_{\perp} = \left(0, *, \frac{\zeta_{\parallel}-\zeta_{\perp}}{2} r_0\omega \sin 2\theta\right)$$

$$\overline{F}_{\omega,z} = \frac{\zeta_{\parallel}-\zeta_{\perp}}{2} r_0\omega \cdot L \cdot \sin 2\theta$$

- pure translation with $\omega = 0$.

$$\underline{V} = (0, 0, V_0)$$

$$\underline{V}_{\parallel} = (0, *, V_0 \sin^2\theta)$$

$$V_{\perp} = (0, * \; V_0 \cos^2\theta).$$

$$f(s=0) = \zeta_{\parallel} V_{\parallel} + \zeta_{\perp} \underline{V_{\perp}}$$

$$= V_0 (0, *, \zeta_{\parallel} \sin^2\theta + \zeta_{\perp} \cos^2\theta).$$

$$\overline{f_{\text{trans},z}} = L \cdot V_0 (\zeta_{\parallel} \sin^2\theta + \zeta_{\perp} \cos^2\theta) + 6\pi \eta a V_0$$

- force balance

$$V_0 = \frac{(\zeta_{\perp} - \zeta_{\parallel}) r_0 \sin 2\theta \cdot \omega}{\zeta_{\parallel} \sin^2\theta + \zeta_{\perp} \cos^2\theta + 6\pi\eta a / L}$$

- $\omega \neq \omega_0$,

Counter-rotation of cell

- torque balance

$$0 = \underline{T}_{flag} + \underline{T}_{cell}$$

$$\underline{T}_{cell} = 8\pi\eta\, a^3(\omega - \omega_0)\, \underline{e}_z .$$

$$\underline{T}_{flag} = \int_0^L \underline{f}(s) \times \underline{r}(s)\, ds.$$

$$0 = 8\pi\eta\, a^3(\omega - \omega_0) +$$

$$\omega \cdot L r_0^2 \cdot g\left(\xi_{\parallel}, \xi_{\perp}, \theta, \frac{6\pi\eta a}{L}\right)$$

$$\omega = \omega_0 \frac{1}{1 + \frac{L r_0^2}{8\pi\eta a^3}\, g\left(\xi_{\parallel}/\xi_{\perp}, \theta, \frac{\eta a}{L}\right)},$$

Vogel Ch. 4

- At large Re, viscosity negligible: Ideal fluid def'd as $\eta = 0$.
- In ideal fluids, Bernoulli's principle holds.

$$\rho \frac{Dv}{Dt} = -\nabla p.$$

mass × acceleration   pressure force

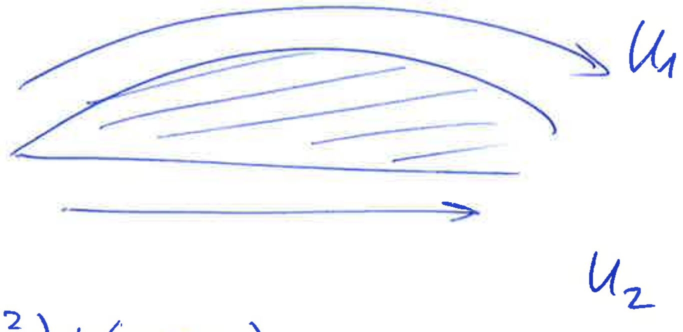
$dt = dl / v$

$$\rho \frac{dv}{dt} = \rho \frac{dv \cdot v}{dl} = -\frac{dp}{dl}$$

$$\frac{\rho}{2} v^2 + p = \text{const}$$

→ generalization to vector case: Euler equation (LL).

- Why does an airplane fly?

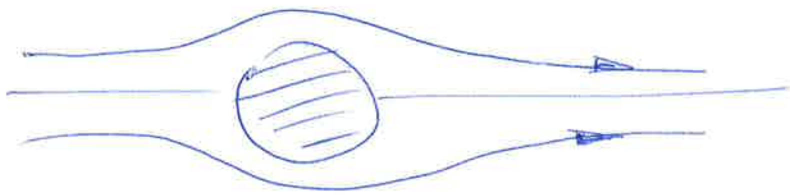


$$\frac{\rho}{2}(u_1^2 - u_2^2) + (p_1 - p_2) = 0.$$

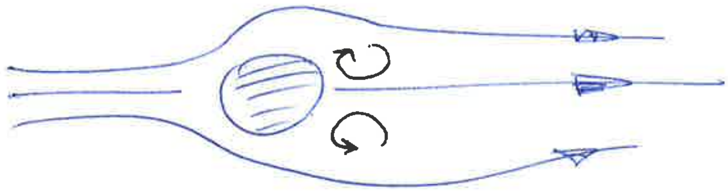
$u_1 > u_2 \Rightarrow p_1 < p_2 \Rightarrow$ lift.

- close to boundary: viscosity not negligible
→ boundary layer, turbulence

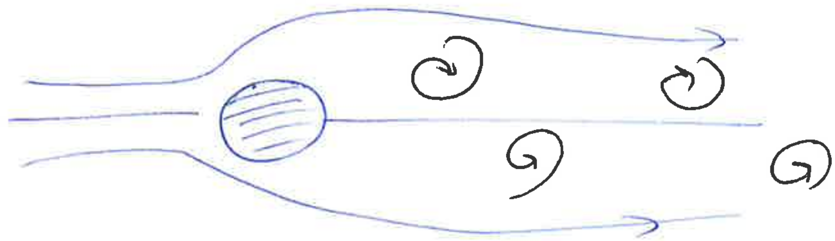
Creeping flow ($Re < 10$).



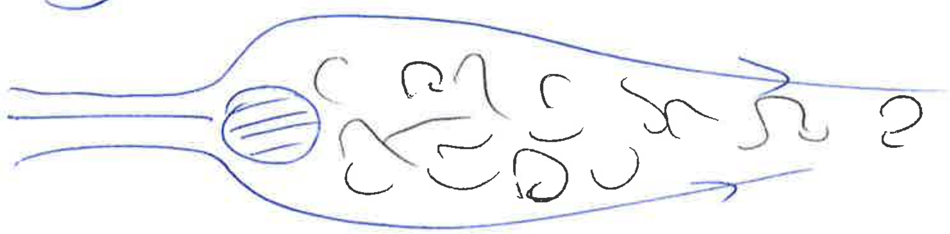
Attached vortices ($10 < Re < 40$).



Van-Karman vortex trail ($40 < Re < 2 \cdot 10^5$)



Fully turbulent wake ($Re > 2 \cdot 10^5$)



Shock waves ....

S Vogel: Life Moving fluids

Conservation of circularity

$\eta = 0$: ideal fluid.

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\nabla p.$$

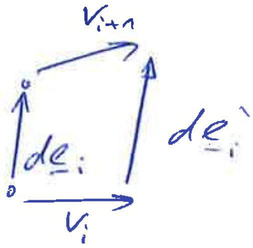
$$\oint_C \underline{v}\cdot d\underline{\ell} = \underbrace{\underline{\nabla}\times\underline{v}}_{\text{vorticity}}\cdot d\underline{A} \equiv \text{circularity}.$$

$$\frac{d}{dt}\oint_{C(t)} \underline{v}\cdot d\underline{\ell}$$

– $\underline{v}$ changes

– C changes.

$$= \frac{d}{dt}\oint_{C(t_0)} \underline{v}\cdot d\underline{\ell} + \underbrace{\oint \underline{v}\cdot \frac{d}{dt}\, d\underline{\ell}}_{\oint \frac{1}{2}\underline{\nabla}|\underline{v}|^2 d\underline{\ell} \,=\, 0}$$



$v_\beta \frac{d}{dt} d\ell_\beta = v_\beta \partial_\alpha v_\beta \ell_\alpha$

integral of total differential

(change in v and dA cancel)

$$= \frac{d}{dt}(\underline{\nabla}\times\underline{v})\cdot d\underline{A}$$

$$= \underline{\nabla}\times\left(\frac{d}{dt}\underline{v}\right)\cdot d\underline{A}$$

$$= \underline{\nabla}\times\left(-\frac{1}{\rho}\nabla p\right)\cdot d\underline{A}$$

$$= \underline{\underline{0}}.$$

Thomson's law.

Navier-Stokes (Newtonian fluid, incompressible).

$$\rho\left[\frac{\partial v}{\partial t} + v\cdot(\underline{\nabla}\otimes\underline{v})\right] = -\nabla p + \eta\nabla^2\underline{v}$$

vector identity

$$\underline{v}\,(\underline{\nabla}\otimes\underline{v}) = \frac{1}{2}\,\underline{\nabla}\,|\underline{v}|^2 - \underline{v}\times(\underline{\nabla}\times\underline{v})$$

Take $\underline{\nabla}\times$ on both sides.

$$\rho\left[\frac{\partial \omega}{\partial t} - \underline{\nabla}\times(\underline{v}\times\underline{\omega})\right] = 0 + \eta\nabla^2\underline{\omega}$$

another vector identity

$$\underline{\nabla}\times(\underline{v}\times\underline{\omega}) = -\underline{v}\cdot(\underline{\nabla}\otimes\underline{\omega}) + \underline{\omega}\cdot(\underline{\nabla}\cdot\underline{v}).$$

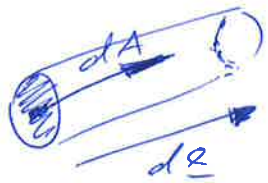
using $\underline{\nabla}\cdot\underline{v} = 0$, $\underline{\nabla}\cdot\underline{\omega} = 0$.

$$\frac{D\underline{\omega}}{Dt} = \underbrace{\underline{\omega}\cdot(\underline{\nabla}\otimes\underline{v})}_{\text{change of vorticity perpendicular to streamline}} + \frac{\eta}{\rho}\nabla^2\omega.$$

Batchelor 5.2

Revisit Thomson's law.

$$0 = \frac{D}{Dt}\, \underline{\omega} \cdot d\underline{A}$$



$$dV = d\underline{A} \cdot d\underline{\ell}.$$

$$0 = \frac{D}{Dt}\, dV = \frac{D}{Dt}\left(d\underline{A} \cdot d\underline{\ell}\right)$$

$$= \left(\frac{D}{Dt}\, d\underline{A}\right) \cdot d\underline{\ell} + \underbrace{\left(\frac{D}{Dt}\, d\underline{\ell}\right)}_{d\underline{\ell} \cdot (\underline{\nabla} \otimes \underline{v})} \cdot d\underline{A}$$

Must hold for all $d\underline{\ell}$

$$\frac{D}{Dt}\, d\underline{A} = -(\underline{\nabla} \otimes \underline{v}) \cdot d\underline{A}.$$

$$\frac{D}{Dt}\, \underline{\omega} \cdot d\underline{A} = \underline{\omega} \cdot (\underline{\nabla} \otimes \underline{v}) \cdot d\underline{A} + \frac{\eta}{\rho}\left(\nabla^2 \underline{\omega}\right) \cdot d\underline{A}$$

$$- \underline{\omega}\, (\underline{\nabla} \otimes \underline{v}) \cdot d\underline{A}$$

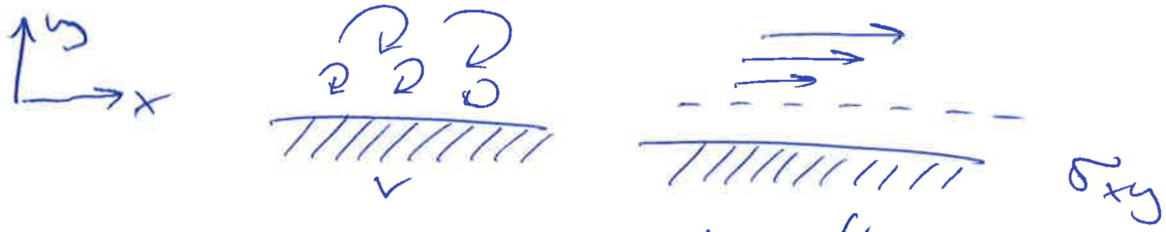
$$= \frac{\eta}{\rho}\left(\nabla^2 \underline{\omega}\right) \cdot d\underline{A}$$

$$= 0 \quad \text{for } \eta = 0.$$

The logarithmic velocity profile

LL §42

Re >> 1



$\langle v \rangle = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}$

- σ_{xy} : $\frac{N}{m^2} = \frac{kg \cdot m/s}{s \cdot m^2}$ = momentum flux.

$v^* = \sqrt{\sigma_{xy}/\rho}$

- $u = u(y)$

$\frac{\partial u}{\partial y} = f(\sigma, \rho, y) = \frac{1}{\varkappa} \frac{v^*}{y}$, $\varkappa = 0.4$.

$u = \frac{v_*}{\varkappa} \ln(y/c)$

What is c?

- Viscous boundary layer.

$Re = \rho \frac{v_* y_0}{\eta}$

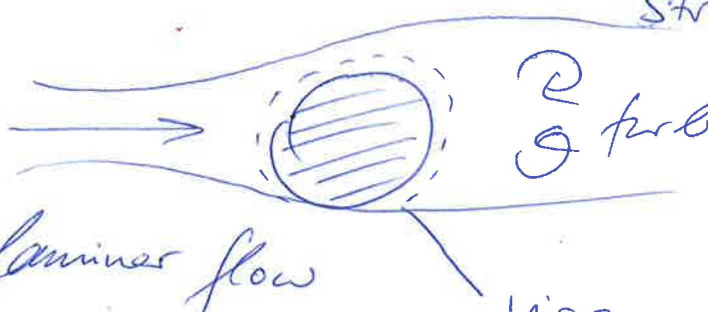
$Re \sim 1 \Rightarrow y_0 \sim \frac{\eta}{\rho v_*}$

$y \ll y_0$: $u = \frac{\sigma}{\eta} y = \frac{\rho v_*^2}{\eta} y \Rightarrow u \sim v_*$ for $y \sim y_0$

- Matching near and far-field

$u = \frac{v_*}{\varkappa} \ln\left(\frac{y}{y_0}\right)$ ≡ log. velocity profile

Turbulent flow around a body



$\underline{\nabla} \times \underline{V} \neq 0.$

laminar flow
=
potential flow
$\underline{\nabla} \times \underline{V} = 0$

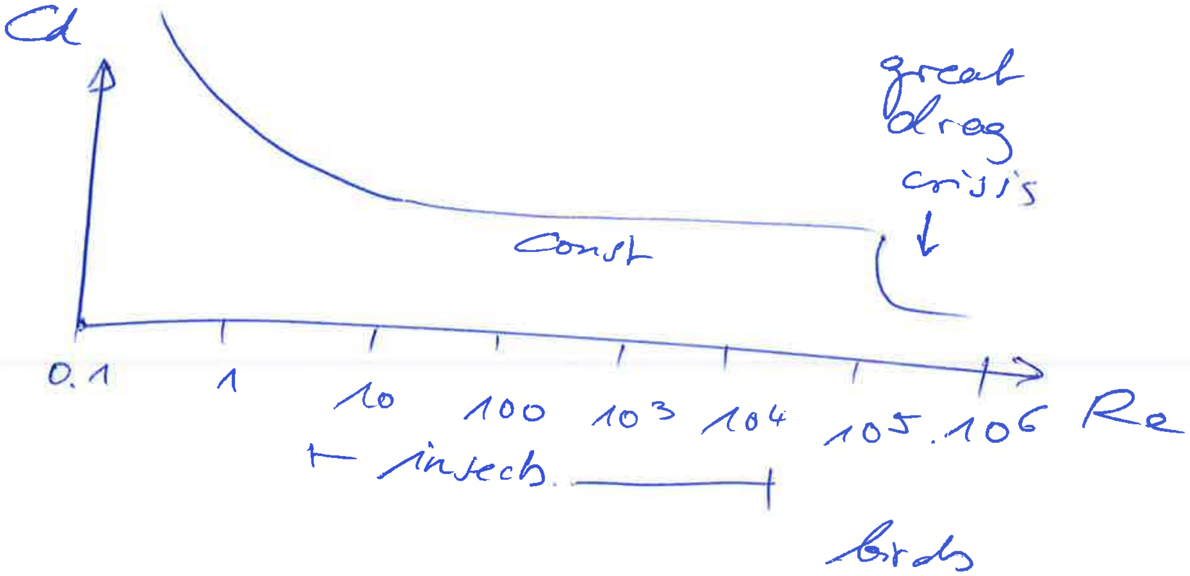
zero vorticity

$$F = \text{const.}\ \rho\, u^2 \cdot L^2 \qquad Re \gg 1.$$

$$= \text{const}\ \eta \cdot u \cdot L \qquad Re \ll 1.$$

Drag coefficients and the Reynolds number.

$$F_{drag} = \left(\frac{1}{2}\,\rho A u^2\right) \cdot Cd(Re).$$



skin friction dominates.

pressure drag:
fluid accelerated to flow around body; [illegible] not returned but dissipated [illegible] [illegible]

turbulence invades boundary layer.
separation line moves forward.

Energy dissipation.

↓ energy input

large eddies  $Re_\ell = \rho \frac{\Delta u \cdot \ell}{\eta}$

↓ energy flow with rate ε

smaller eddies 

↓ viscous dissipation.

Kolmogorof lengthscale: $\lambda^4 = \frac{(\eta/\rho)^3}{\varepsilon}$ by dimensions

$$Re_\lambda = \rho \frac{\Delta u_\lambda \cdot \lambda}{\eta} \sim 1.$$

rate of energy flow

$$\varepsilon \sim \frac{(\Delta u_\ell)^3}{\ell}$$

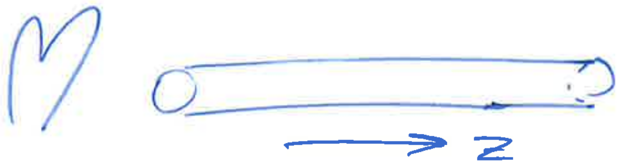
$\left[\frac{N}{kg \cdot s}\right]$

by dimensions.

ε constant across scales.

$\Delta u_\ell \sim \ell^{1/3} \equiv$ self-similarity.

$$\rho \frac{\partial v}{\partial t} = -\nabla p + \eta \nabla^2 v \qquad (*)$$



$$\frac{\partial p}{\partial z} = p_0 \exp i\omega_0 t \, .$$

Ansatz $\underline{v} = u(r) \exp i\omega_0 t \; \underline{e}_z$.

(*) gives:

$$\rho \, i\omega_0 t \, u = -p_0 + \eta \left(u_{rr} + \frac{1}{r} u_r \right)$$ Laplace op. polar coords

$\longrightarrow$

$$u(r) = \frac{i p_0}{\rho \omega_0} \left(1 - \frac{J_0(i^{3/2} \alpha \, r/R)}{J_0(i^{3/2} \alpha)} \right)$$

$$\alpha = \sqrt{\frac{\omega_0 \rho}{\eta}} R \equiv \text{Womersley number}$$

= 15 aorta
5 arteries
0.05 arterioles
0.005 capillaries

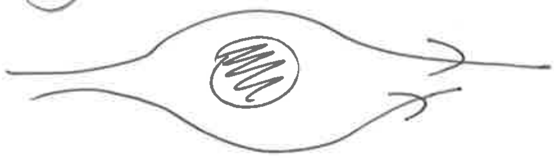
$$\text{flux} = 2\pi \int_0^R u(r) \, r \, dr =$$

$$\frac{-i\pi p_0 R^4}{\eta} \frac{\exp i\omega_0 t}{\alpha^2} \frac{J_2(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}$$

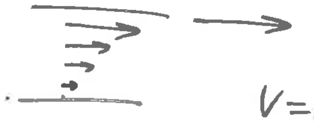
$$= \begin{cases} \dfrac{\pi p_0 R^4}{8\eta} & |\alpha| \ll 1 \\[2ex] \dfrac{\pi p_0 R^2}{\omega_0 \rho} & |\alpha| \gg 1 \end{cases}$$

$Re = 0$ viscid flow.

(creeping flow, Stokes flow)



laminar. = stream.lines locally parallel

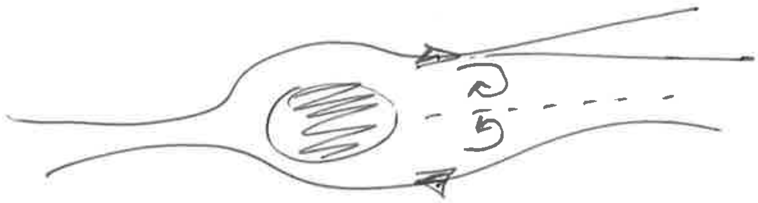


$v = \begin{pmatrix} uy/h \\ 0 \\ 0 \end{pmatrix}, \; \omega = \begin{pmatrix} 0 \\ 0 \\ u/h \end{pmatrix}$

2000 - 5000.

turbulent

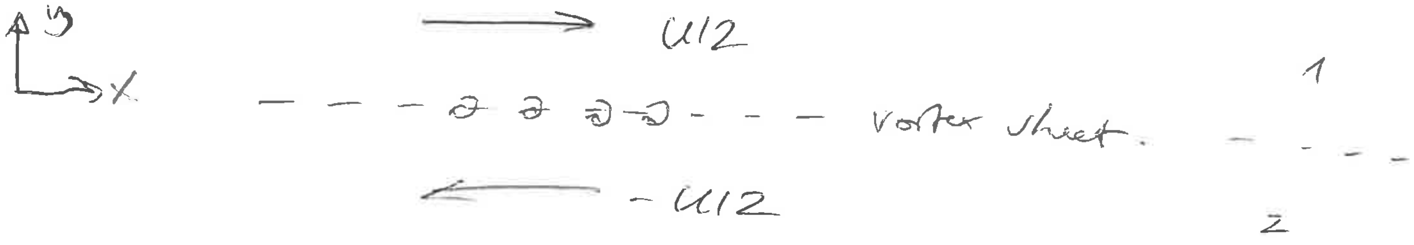
- chaotic (⇒ mixing).
- unsteady vortices.
- $\omega \neq 0$.



$Re = \infty$: ideal fluid: inviscid flow.

- steady / unsteady.

Instabilities of flows



- Ideal fluid ($\nu = 0$): infinitesimal thin boundary layer
- Q: Will the boundary layer remain straight?

  $h = h(x,t)$ = height profile

- In the upper/lower half-space flow $\rightarrow$ laminar and irrotational $\omega = 0$.

  potential flow $\quad v = \nabla \varphi$.

  $$\varphi_1 = +\frac{U}{2}x + \phi_1$$

  $$\varphi_2 = -\frac{U}{2}x + \phi_2 .$$

- h, ϕ_1, ϕ_2 coupled:
- $$v_y^{(1)} = \frac{\partial \phi_1}{\partial y}\Big|_{y=0} = \frac{Dh}{Dt}\Big|_{y=0} \approx \frac{\partial h}{\partial t} - \frac{1}{2}U\frac{\partial h}{\partial x} \qquad (1)$$

  $$v_y^{(2)} = \frac{\partial \phi_2}{\partial y}\Big|_{y=0} = \frac{Dh}{Dt}\Big|_{y=0} \approx \frac{\partial h}{\partial t} + \frac{1}{2}U\frac{\partial h}{\partial x}. \qquad (2)$$

- 2^{nd} equation from pressure difference.
- Generalize Bernoulli's principle
  - $\frac{Dv}{Dt} = -\frac{1}{\rho} p.$ ≡ Navier-Stokes for incompressible, ideal fluid
    $= \frac{\partial v}{\partial t} + \underline{v} \cdot (\underline{\nabla} \otimes \underline{v})$
  - Vector identity

    $$\frac{1}{2} \nabla (\underline{v} \cdot \underline{v}) = \underline{v} \cdot \underline{\nabla} \underline{v} + \underline{v} \times \underline{\omega}$$

  - $v = \nabla \psi, \quad \omega = 0.$
  - $$\nabla \left(\frac{\partial \psi}{\partial t} + \frac{1}{2} |v|^2 + \frac{p}{\rho} \right) = 0.$$

[Batchelor (6.2.5)]

$$\longrightarrow \frac{\partial \psi}{\partial t} + \frac{1}{2} |v|^2 + \frac{p}{\rho} = \text{const.}$$

(N.B. possibly do Gauge trf. on ψ)

- $$p_i = p_0 - \rho \left(\frac{\partial \psi_i}{\partial t} + \frac{1}{2} |v|^2 \right).$$
- no surface tension

$$0 = \left(\frac{p_1 - p_2}{\rho} \right)\Big|_{y=h} = \underbrace{\frac{\partial \psi_2}{\partial t} - \frac{\partial \psi_1}{\partial t}}_{\frac{\partial \phi_2}{\partial t} - \frac{\partial \phi_1}{\partial t}} + \underbrace{\frac{1}{2} \left(|v_1|^2 - |v_2|^2 \right)}_{\approx \frac{1}{2} U \left(\frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_1}{\partial x} \right)} \quad (3)$$

- Ansatz: $h = H(t) \exp ikx,$
  $\phi_1 \sim h \exp -ky, \quad \phi_2 \sim -h \exp +ky. \quad (\nabla^2 \phi = 0).$

Ansatz:

$$h = ff(t)\ \exp ikx$$
$$\phi_1 \sim h\ \exp -ky$$
$$\phi_2 \sim h\ \exp +ky.$$

(Motivation $\underline{\nabla} \cdot \underline{v} = 0 \Rightarrow \nabla^2 \phi = 0$).

(1)+(2): $-k(\phi_1 - \phi_2) = 2\dot{h}$
(1)−(2): $-k(\phi_1 + \phi_2) = u h ik$

(3): $\frac{\partial}{\partial t}\underbrace{(\phi_1 - \phi_2)}_{-\frac{2}{k}\dot{h}} = \frac{1}{2} u ik \underbrace{(\phi_1 + \phi_2)}_{-u i h}$

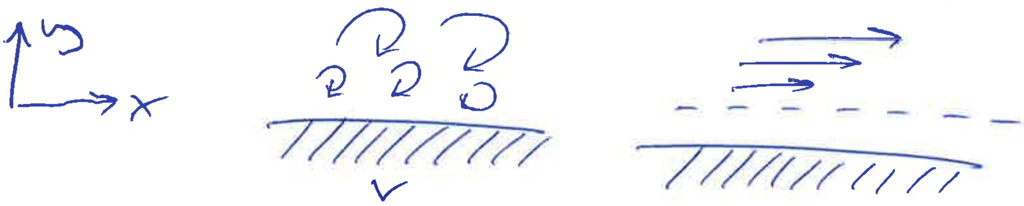
$$\ddot{h} = \frac{1}{4}(ku)^2 h.$$
$$h \sim \exp \pm \frac{ku}{2} t.$$

⇒ unstable solution
⇒ undulations of vortex sheet grow.

The logarithmic velocity profile

LL §42

$Re \gg 1$



σ_{xy}

$\langle v \rangle = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}$

- $\sigma_{xy} : \frac{N}{m^2} = \frac{kg \cdot m/s}{s \cdot m^2}$ = momentum flux.

  $v^* = \sqrt{\sigma_{xy}/\rho}$

- $u = u(y)$

  $\frac{\partial u}{\partial y} = f(\sigma, \rho, y) = \frac{1}{\varkappa} \frac{v^*}{y}$, $\varkappa = 0.4$.

  $u = \frac{v_*}{\varkappa} \ln(y\cancel{/}c)$

  What is c?

- Viscous boundary layer.

  $Re = \rho \frac{v_* y_0}{\eta}$

  $Re \sim 1 \Rightarrow y_0 \sim \frac{\eta}{\rho v_*}$

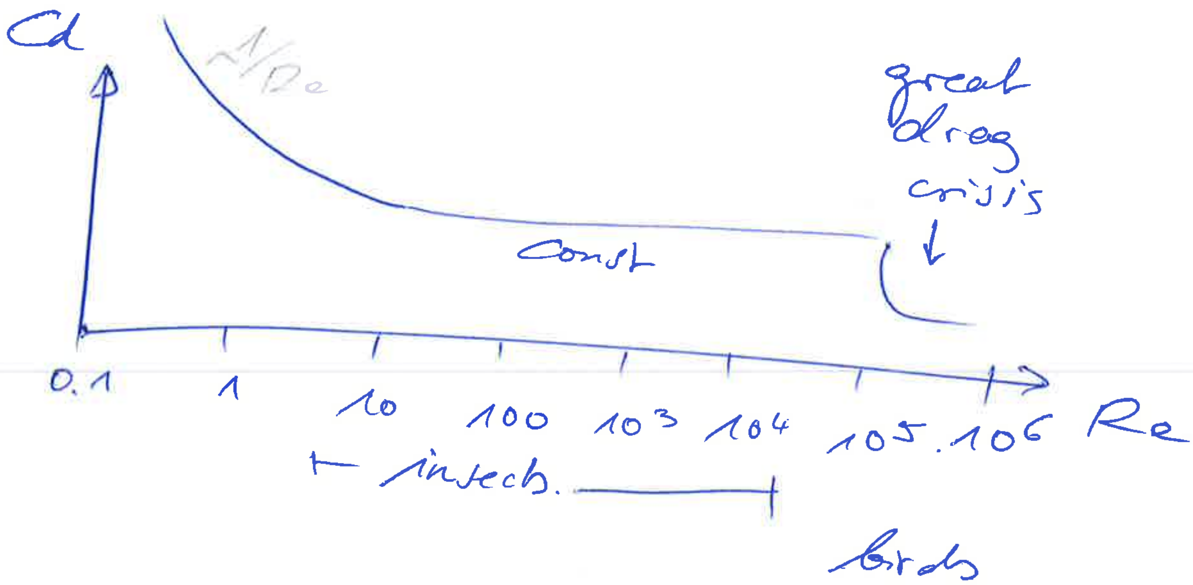
  $y \ll y_0$: $u = \frac{\sigma}{\eta} y = \frac{\rho v_*^2}{\eta} y \Rightarrow u \sim v_*$ for $y \sim y_0$

- Matching near and far-field

  $u = \frac{v_*}{\varkappa} \ln\left(\frac{y}{y_0}\right) \equiv$ log. velocity profile

Drag coefficients and the Reynolds number.

$$F_{drag} = \left(\frac{1}{2}\rho A u^2\right) \cdot Cd(Re).$$



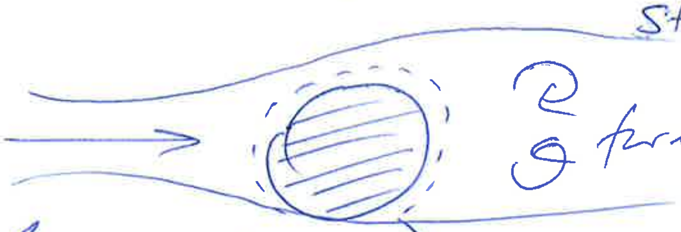
skin friction dominates.

pressure drag:
fluid accelerated to flow around body
[illegible]

turbulence invades boundary layer.
separation line moves forward.

Turbulent flow around a body

streamlines



turbulence $\underline{\nabla} \times \underline{v} \neq 0$.

Wake (Nachlauf).

laminar flow
=
potential flow
$\underline{\nabla} \times \underline{v} = 0$

zero vorticity

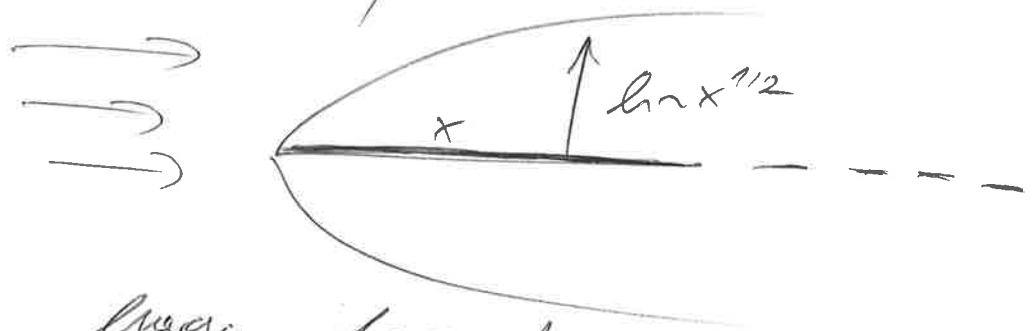
viscous boundary layer

$$F = \text{const.}\ \rho\, u^2 \cdot L^2 . \qquad Re \gg 1 .$$

$$= \text{const}\ \eta \cdot u \cdot L \qquad Re \ll 1 .$$

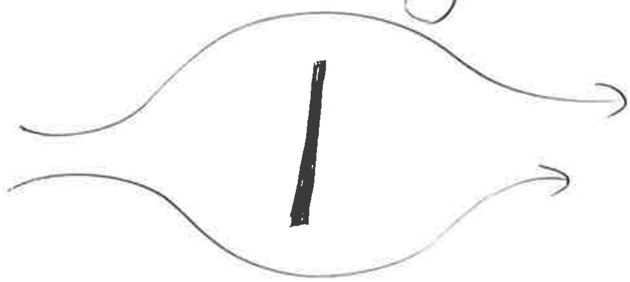
Two contributions to hydrodynamic drag at high Re.

- skin friction



energy dissipation in viscous boundary layer.
$\rightarrow$ wetted area important.

- pressure drag.



- at $Re=0$:
  flow antisymmetric wrt front/back - reflection
- at high Re:
  difference in kinetic pressure $\frac{v^2}{\rho}$ between front and back.

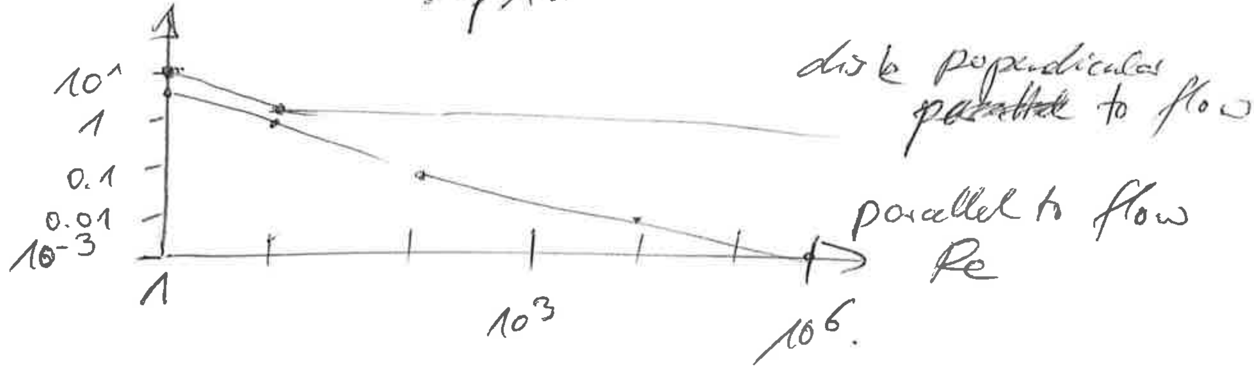
front
$\downarrow$
energy invested to accelerate fluid

back
$\downarrow$
kinetic energy dissipated due to viscosity

$\longrightarrow$ Cross-sectional area important.

Experimental example.

$$C_d = \frac{2 F_{drag}}{\rho A U^2}$$



- disk - perpendicular to flow:
  pressure drag dominates.

- disk parallel to flow.
  skin friction dominates.

$$F_d \sim U \Rightarrow C_d \sim \frac{1}{Re}.$$

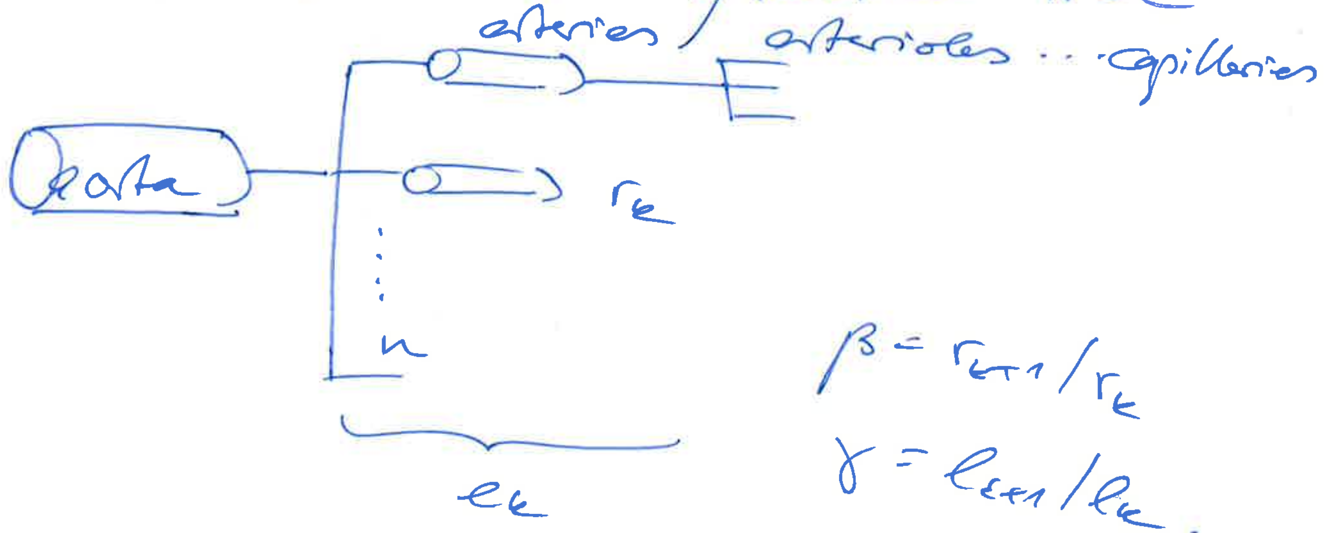
West et. al.:

Metabolic Scaling.

Metabolism $B \sim M^{3/4}$.

N.B. ongoing controversy.

Vascular network = fractal tree



$\beta = r_{k+1}/r_k$

$\gamma = l_{k+1}/l_k$.

$N_c = n^N = 10^{10}$ for humans.

Metabolism

$B \sim$ blood flow $\cancel{\neq N_1 \mathcal{J}_k}$ $\mathcal{J}_0$

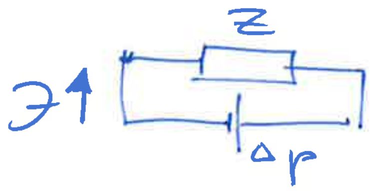
$= N_c \mathcal{J}_c$

$= N_c \, \pi \, r_c^2 \, \bar{u}_c$

$\sim N_c = n N \equiv$ total number of capillaries

blood pressure

$\Delta p = \mathcal{J} \cdot Z$. (electric analogy)



Cardiac output

$W = \mathcal{J} \cdot \Delta p$.

$$\rho \frac{\partial v}{\partial t} = -\nabla p + \eta \nabla^2 v \qquad (*)$$

$$\frac{\partial p}{\partial z} = p_0 \exp i\omega_0 t \, .$$

Ansatz $\quad \underline{v} = u(r) \exp i\omega_0 t \; \underline{e}_z \, .$

$(*)$ gives:

Laplace op. polar coords

$$\rho \, i\omega_0 t \, u = -p_0 + \eta \left(u_{rr} + \frac{1}{r} u_r \right)$$

$\longrightarrow$

$$u(r) = \frac{i p_0}{\rho \omega_0} \left(1 - \frac{J_0(i^{3/2} \alpha \, r/R)}{J_0(i^{3/2} \alpha \;)} \right)$$

$$\alpha = \sqrt{\frac{\omega_0 \rho}{\eta}} R \equiv \text{Womersley number}$$

= 15 aorta
5 arteries
0.05 arterioles
0.005 capillaries

$$\text{flux} = 2\pi \int_0^R u(r) \, r \, dr =$$

$$\frac{-i\pi p_0 R^4}{\eta} \frac{\exp i\omega_0 t}{\alpha^2} \frac{J_2(i^{3/2}\alpha)}{J_0(i^{3/2}\alpha)}$$

$$= \begin{cases} \dfrac{\pi \, p_0 R^4}{8\eta} & |\alpha| \ll 1 \\[2ex] \dfrac{\pi \, p_0 R^2}{\omega_0 \rho} & |\alpha| \gg 1 \end{cases}$$

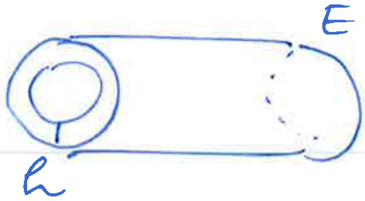
$$\text{impedance} = Z = \frac{\Delta p}{\text{flux}}, \quad \Delta p = \ell \cdot \frac{\partial p}{\partial z}.$$

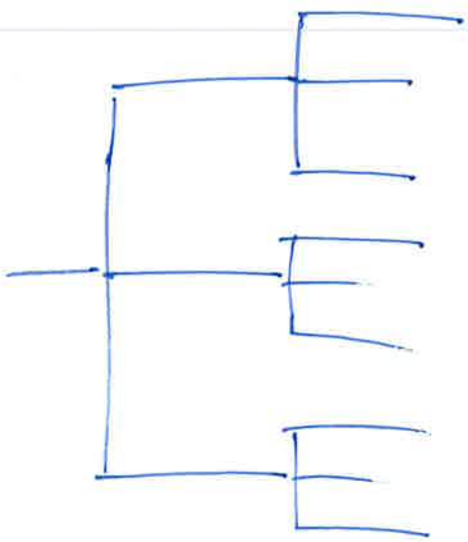
$$|Z| = \frac{8\eta\ell}{\pi R^4} \qquad |\alpha| \ll 1.$$

$\equiv$ Poiseuille flow

$$|Z| = \frac{\rho\omega_0\ell}{\pi R^2} \qquad |\alpha| \gg 1.$$

N.B. elasticity of vessel walls gives correction





. . . E E E

$$|Z| = \frac{\rho c_0}{\pi R^2}$$

$$c_0 = \sqrt{\frac{Eh}{2\rho R}}$$

aorta $\alpha = 15$ arteries $\alpha = 5$ arterioles $\alpha = 0.05$ capillaries $\alpha = 0.005$.

inertial effects dominate

pulsatile flow

viscous effects dominate

non-pulsatile flow

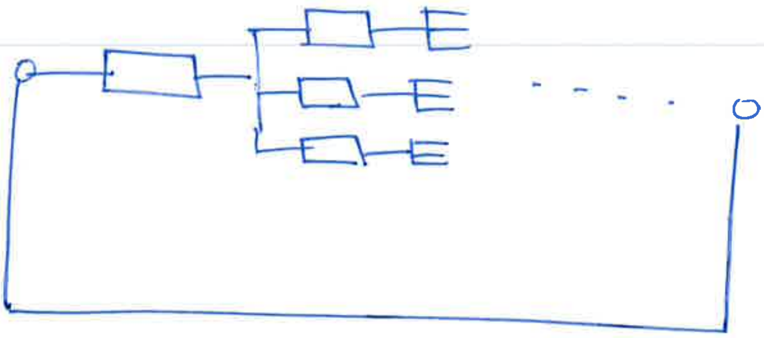
Non-pulsatile flow.

Viscous resistance (impedance)

$$R_k = \frac{8 \eta l_k}{\pi r_k^4}$$

resistance of network.

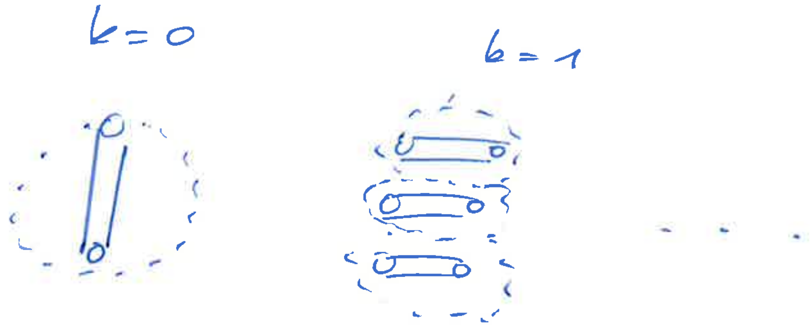
$$Z = \sum_{k=0}^{N} \frac{R_k}{N_k} \approx \frac{R_c}{}$$



(electrical analogy)

The concept of Lerice volumes:

$$V_{tot} = \frac{4}{3}\pi \cdot \left(\frac{\ell_k}{2}\right)^3 N_k$$



$$\Rightarrow \quad \gamma^3 = \frac{\ell_{k+1}^3}{\ell_k^3} = \frac{N_k}{N_{k+1}} = \frac{1}{n}$$

$$\boxed{\gamma \sim n^{-1/3}}$$

Impedance matching.

- Non-pulsatile flow

$$R_k = \frac{1}{n} R_{k+1}$$

$$\Rightarrow \frac{\ell_k}{r_k^4} = \frac{1}{n} \frac{\ell_{k+1}}{r_{k+1}^4}$$

$$\Rightarrow \beta^4 = \gamma / n \Rightarrow \beta = n^{-1/3}.$$

Murray's law: $n \beta^3 =$ const. across levels

- pulsatile flow

$$R_k = \frac{1}{n} R_{k+1}$$

$$\frac{1}{r_k^2} = \frac{1}{n} \frac{1}{r_{k+1}^2}$$

$$\Rightarrow \beta = n^{-1/2}.$$

pulsatile
- number of levels changes with M.

non-pulsatile.
- number of levels const.

$$\alpha = \frac{1}{n\gamma\beta^2} \sim n^{1/3}.$$

$$M \sim V_{blood} \sim N_{tot}\, \pi r_c^2 l_c \left(1 + \alpha^2 + \alpha^2 + \dots \alpha^N\right)$$

$$\sim N_{tot}^{4/3}.$$

$$B \sim N_{tot}$$

$$\Rightarrow \boxed{B \sim M^{3/4}.}$$

N.B.

refinement: l_c not constant

$$N_{tot}\, l_c^3 = N_{tot}^{4/3}\, l_c$$

$$\longrightarrow l_c \sim N_{tot}^{1/6}$$

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