Linear stability analysis

Given: nonlinear system

$$\dot{\underline{x}} = f(\underline{x}) \qquad (*)$$

with fixed point $f(\underline{x}_0) = 0$.

Q: dynamics of small perturbation $\underline{y} = \underline{x} - \underline{x}_0$.

$\Rightarrow$ Consider linearized system

$$\dot{\underline{y}} = \underbrace{\underline{\underline{J}}\,\underline{y}}_{\text{leading order term of Taylor expansion of } (*)} \quad \text{with} \quad \underline{\underline{J}} = Df|_{\underline{x}=\underline{x}_0} \equiv \text{Jacobian}$$

A: stability of $\underline{x}_0$ determined by eigenvalues of $\underline{\underline{J}}$. (expt in non-generic borderline cases)

Classification of linear systems.

- $n = 1$: $\underline{\underline{J}} = \lambda$

$$y(t) = y(0) \exp \lambda t.$$

$\lambda > 0$: unstable

$\lambda < 0$: stable

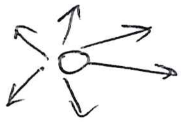
$\lambda = 0$: higher-order nonlinearities matter

- $n = 2$: $\exists$ $\underline{\underline{A}}$ has 2 eigenvalues λ_1, λ_2 with eigenvectors $\underline{v}_1, \underline{v}_2$.

$\Rightarrow$ think of $\underline{v}_i$ as perturbation modes

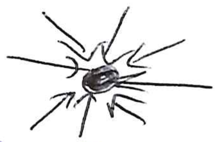
$$y(t) = \alpha_1 \underline{v}_1 \cdot \exp \lambda_1 t + \alpha_2 v_2 \exp \lambda_2 t.$$

Case Ia: $\lambda_1, \lambda_2 > 0$: unstable node

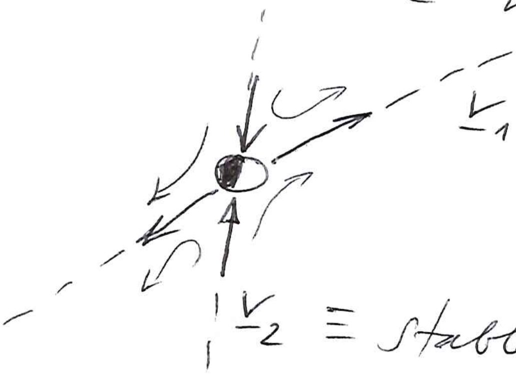


Case Ib: $\lambda_1, \lambda_2 < 0$: stable node

$|\lambda_1| > |\lambda_2|$
fast slow eigendirection
(only dimensionless quantities can be small or large)



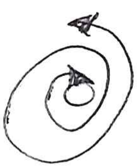
Case II: $\lambda_1 > 0$, $\lambda_2 < 0$: saddle node



$\underline{v}_1 \equiv$ unstable manifold

$\underline{v}_2 \equiv$ stable manifold.

Case III a, $\mathrm{Re}\,\lambda_i > 0$, $\mathrm{Im}\,\lambda_i \neq 0$: unstable spiral



Case III b, $\mathrm{Im}\,\lambda_i \neq 0$, $\mathrm{Re}\,\lambda_i < 0$: stable spiral

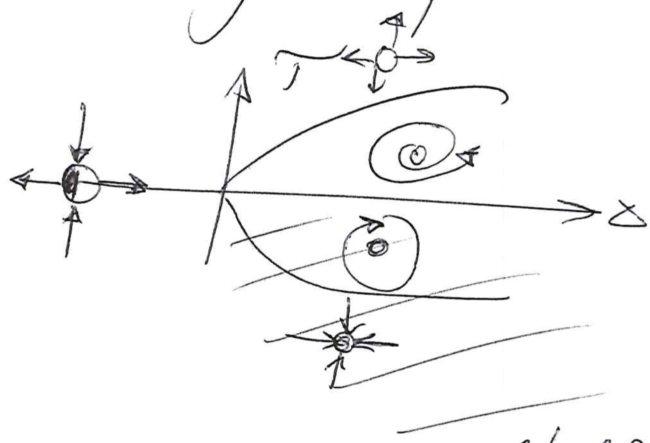
Instead of the eigenvalues use trace and determinant as invariant of $\underline{\underline{J}}$.

$$\mathrm{tr}\,\underline{\underline{J}} = T = \lambda_1 + \lambda_2 \,, \quad \det \underline{\underline{J}} = \Delta = \lambda_1 \cdot \lambda_2$$

or

$$\lambda_{1,2} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4\Delta}\right).$$

$\Rightarrow$ above case yield phase diagram



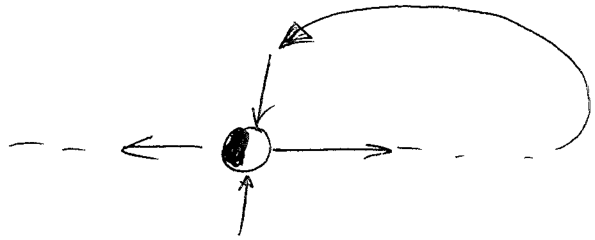
stability quadrant.

- general case: $n > 2$.

$\underline{x}_0$ stable $\Leftrightarrow$ $\mathrm{Re}\,\lambda_i < 0$ for all eigenvalues of $\underline{\underline{J}}$
$\Leftrightarrow$ $\mathrm{tr}\,\underline{\underline{J}}^k < 0$, for $k = 1, \dots, n$
$\equiv$ Routh-Hurwitz-criterion.

If $\underline{x}_0$ is a saddle node with $\mathrm{Re}\,\lambda_1, \dots, \mathrm{Re}\,\lambda_k < 0$, $\mathrm{Re}\,\lambda_{k+1}, \dots, \mathrm{Re}\,\lambda_n > 0$.

But watch out for homoclinic orbits.



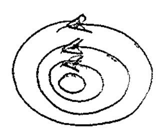
3. Definitions of stability:

- Lyapunov stable = linearly stable
  $\mathrm{Re}\,\lambda_i < 0$.

- asymptotically stable.
  $|\underline{x}(0) - \underline{x}_0| < \varepsilon \Rightarrow \underline{x}(t) \to \underline{x}_0$ for $t \to \infty$

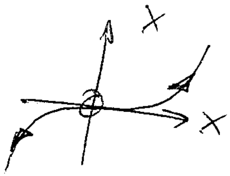
  example: $\dot{\theta} = \omega_0 (1 + \sin\theta)$. 

- neutrally stable:
  $\equiv$ perturbations neither grow nor decay

  example: $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1$  $r^2 = x_1^2 + x_2^2$ $\equiv$ invariant.

  $\equiv$ center $\equiv$ non-hyperbolic f.p.

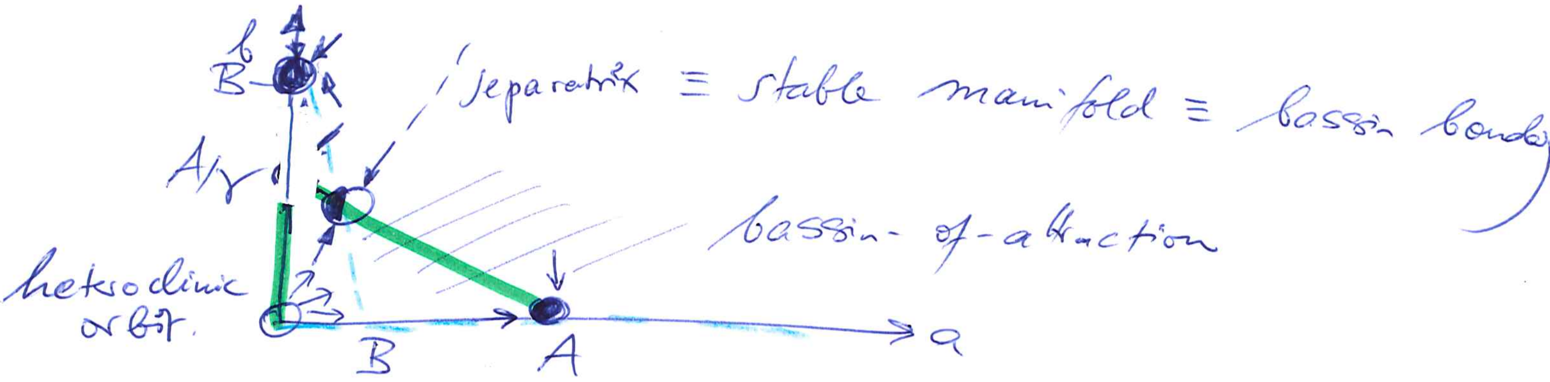
  Counterexample: $\dot{x} = x^3$



N.B. Centers only occur typically if the system has conserved quantities (e.g. energy) or special symmetries.

Example : Lotka-Volterra-model of competition

$$T\dot{a} = a(A - a - \gamma b), \quad \gamma B > A > B, \quad \gamma > 1$$
$$T\dot{b} = b(B - b - a).$$



- null clines : $\dot{a} = 0$, $\dot{b} = 0$
- fixed points = intersections of null clines
- stability

$(0,0)$: $\dot{a} = \frac{A}{T} a$, $\dot{b} = \frac{B}{T} b$.

$(A,0)$: $J = \begin{pmatrix} -A & -\gamma \\ -1 & B \end{pmatrix}$

$\left.\begin{array}{l} \Delta = \det J = -AB - \gamma < 0 \\ \mathrm{tr}\, J = -A + B < 0 \end{array}\right\}$ stable

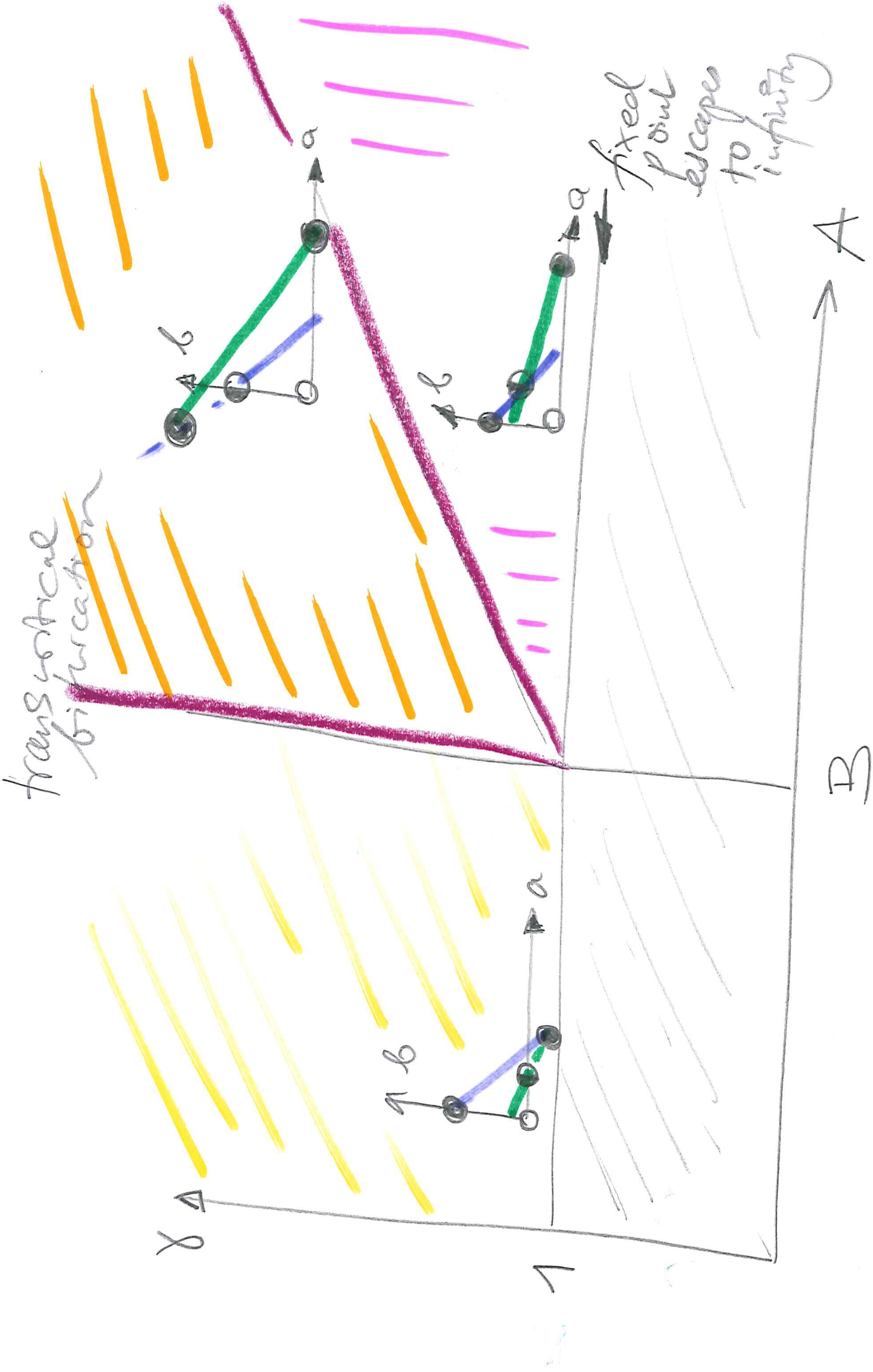
$(0,B)$: exercise

$\frac{1}{\gamma-1}(-A+\gamma B, A-B)$: $J = \frac{1}{\gamma - 1}\begin{pmatrix} A - \gamma B & \gamma(A-\gamma B) \\ -A+B & -A+B \end{pmatrix}$

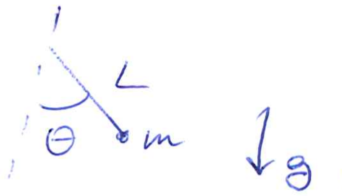
$\Delta = \det J = (A-B)(A-\gamma B)/(\gamma - 1) < 0.$

$\mathrm{tr}\, J = -B < 0.$

$\Rightarrow$ saddle node


transcritical bifurcation
fixed point escape to infinity
A
B
a
b

Example: pendulum



$$\underbrace{-g\cdot m \sin\theta}_{\text{force}} = \underbrace{mL\ddot{\theta}}_{\text{mass} \times \text{acceleration}}$$

$\Rightarrow \quad \ddot{\theta} + \omega_0^2 \sin\theta = 0, \quad \omega = \sqrt{g/L}$

$\Rightarrow \quad \dot{\theta} = v \quad \equiv$ angular velocity

$\dot{v} = -\omega^2 \sin\theta$

fixed points $(0,0)$: $\underline{J} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \Rightarrow$ center?

$(\pi, 0)$: $\underline{J} = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \Rightarrow$ saddle node

How to prove that $(0,0)$ is a center?

Way #1: the system is conservative

$$E = \underbrace{\frac{m}{2}(vL)^2}_{\text{kinetic energy}} - \underbrace{mgL\cos\theta}_{\text{potential energy}} \equiv \text{total energy}$$

$\dot{E} = mvL^2(-\omega^2 \sin\theta) + mgL \sin\theta\, v = 0.$

$E \overset{!}{=} \min$ @ $(0,0)$.

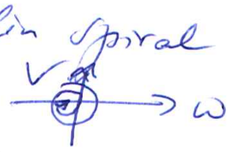
$\Rightarrow$ orbits in vicinity must circle around fixed point (level sets)

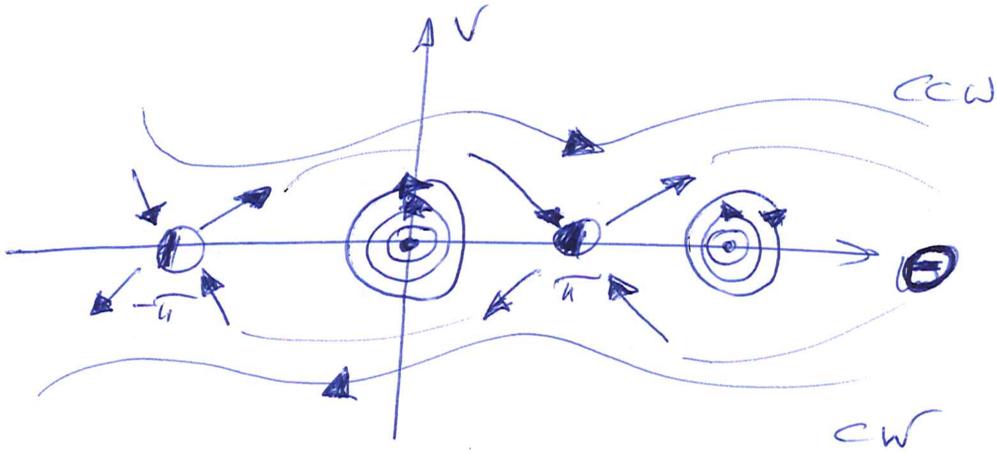
Way #2: the system is reversible.

$t \to -t, \quad v \to -v \equiv$ dyn. invariant.

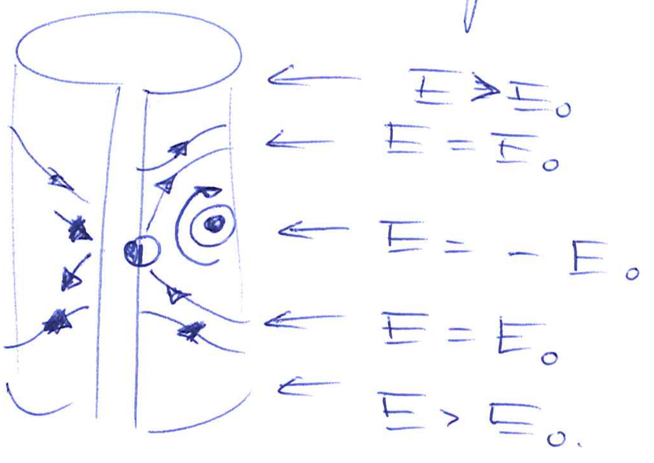
$\Rightarrow$ incompatible with nonlin spiral

$\Rightarrow$ nonlinear center.





$\Downarrow$ wrap to cylinder



$E_0 = mgL$

$\Downarrow$ bend cylinder