2749.

Pattern formation

Dynamics of ~~pattern formation~~
conc. field $c(x,t)$.

Warm-up: diffusion

- conservation equation

$$\frac{\partial}{\partial t} c(x,t) = -\frac{\partial}{\partial x} J.$$

$$J = -D \frac{\partial}{\partial x} c(x,t) \equiv \text{flux}.$$

- refl. b.c.

$$J(x=0) = J(x=L) = 0.$$

- Fourier decomposition

$$c = c_0 + \sum_n c_n \exp \frac{\pi i n x}{L} + \text{c.c.}$$

↓

Phase space

$$\{ c_n,\ n \geq 0 \}$$

- Invariant of motion $c_0(t) = \bar{c}$
  $\Rightarrow$ foliation
- Dispersion relation

$$\frac{\partial}{\partial t} c(x,t) = D \frac{\partial^2}{\partial x^2} c(x,t)$$

$$\partial_t \tilde{c}(q,t) = -D q^2 \tilde{c}(q,t)$$

$$\tilde{c}(q,t) = \sum_n c_n(t)\, \delta\left(q - \frac{\pi i n}{L}\right)$$

$$\Rightarrow \dot{c}_n(t) = -D q^2 c_n(t).$$

⇒ restrict to hyperplane
$c_0(t=0) = \bar{c}$.

⇒
- $c(x) \equiv \bar{c}_0$ ⇝ stable fixedpoint

- perturbation mode
$\exp iqx$ has Ljapunov
exponent Dq^2 $[s^{-1}]$.

A simple Turing system:

- two chemical species,
  concentrations $A(x,t)$, $B(x,t)$.

$$\frac{\partial}{\partial t} A = \alpha_A P(A,B) - \beta_A A + D_A \frac{\partial^2}{\partial x^2} A.$$

$$\frac{\partial}{\partial t} B = \alpha_B P(A,B) - \beta_B B + D_B \frac{\partial^2}{\partial x^2} B.$$

$$P(A,B) = \frac{A^h}{A^h + B^h} \longrightarrow \Theta(A-B)$$

for $h \to \infty$.

$$\Lambda = \frac{\alpha_A \beta_B}{\alpha_B \beta_A} < 1.$$

$$\left.\begin{aligned} \lambda_A &= \sqrt{D_A/\beta_A} \\ \lambda_B &= \sqrt{D_B/\beta_B} \end{aligned}\right\} \equiv \text{two length-scales}$$

$\Rightarrow$ two dim'less parameters.

Homogeneous steady state

$$A(x) = A^*, \quad B(x) = B^*.$$

$$A^* = \Lambda B^*,$$

self-cons. equ. $B^* = \frac{\alpha_B}{\beta_B} P(\Lambda B^*, B^*) = \frac{\alpha_B}{\beta_B} \frac{1}{1+\Lambda^h}$

Linear stability analysis

$$A = A^* + a \exp iqx,$$
$$B = B^* + b \exp iqx.$$

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \underline{\underline{M}}_q \begin{pmatrix} a \\ b \end{pmatrix}.$$

$$\underline{\underline{M}}_0 = \begin{pmatrix} R_{AA} & R_{AB} \\ R_{BA} & R_{BB} \end{pmatrix}, \quad \begin{aligned} R_{AA} &= \partial_A R_A \\ R_{AB} &= \partial_B R_A. \\ R_{BA} &= \partial_A R_B. \\ R_{BB} &= \partial_B R_B. \end{aligned}$$

where.

$$R_A = \alpha_A P - \beta_A A.$$
$$R_B = \alpha_B P - \beta_B B.$$

$$\underline{\underline{M}}_q = \underline{\underline{M}}_0 - \begin{pmatrix} D_A & \\ & D_B \end{pmatrix} q^2.$$

Specifically for our system

$$\partial_A P = \frac{h \beta_A / \alpha_A}{1 + \Delta h} \Rightarrow R_{AA} = \alpha_A \partial_A P - \beta_A$$

$$\partial_B P = - \Delta \partial_A P \Rightarrow R_{BB} = \alpha_B \partial_B P - \beta_B$$

$\Rightarrow$

$$\det \underline{\underline{M}}_q = (\beta_A + D_A q^2)(\beta_B + D_B q^2) - \frac{\beta_A \beta_B \, q^2 h}{1 + \Delta h} \left(\underbrace{\frac{D_B}{\beta_B}}_{\lambda_B^2} - \underbrace{\frac{D_A}{\beta_A}}_{\lambda_A^2} \right)$$

$\Rightarrow \det \underline{\underline{M}}_q < 0$ implies $\boxed{\lambda_B > \lambda_A}$

Spontaneous formation of inhomogeneous patterns requires.

(1) (A^*, B^*) stable for $q=0$.
$\text{tr}\,\underline{\underline{M}}_0 < 0$, $\det \underline{\underline{M}}_0 > 0$.

(ii) (A^*, B^*) unstable for some $q>0$.

Let's write this down:

- $\text{tr}\,\underline{\underline{M}}_0 = R_{AA} + R_{BB} < 0$. (1)

- $\det \underline{\underline{M}}_0 = R_{AA} R_{BB} - R_{AB} R_{BA} > 0$. (2)

- $\text{tr}\,\underline{\underline{M}}_0 < 0 \Rightarrow \text{tr}\,\underline{\underline{M}}_q = \text{tr}\,\underline{\underline{M}}_0 - (D_A + D_B) q^2 < 0$.
$\Rightarrow$ We need $\det \underline{\underline{M}}_q < 0$ for some q.

- $\det \underline{\underline{M}}_q = \det \underline{\underline{M}}_0 - (D_A R_{BB} + D_B R_{AA}) q^2 + D_A D_B q^4$ (3)
(N.B. $\det \underline{\underline{M}}_q = \det \underline{\underline{M}}_{-q}$)

What do we learn? Necessary cond.

- (1) $\Rightarrow$ $R_{AA} < 0$ or $R_{BB} < 0$.
(3) $\Rightarrow$ $R_{AA} > 0$ or $R_{BB} > 0$.

Wlog $R_{AA} > 0$, $R_{BB} < 0$.

$\Downarrow$ self-activating

$\Downarrow$ self-inhibitory.

• (2) $\Rightarrow$ $R_{AB} R_{BA} < 0$

Cross-reaction terms.

$\Rightarrow$ R_{AB} and R_{BA} have opposite sign.

(N.B. not possible in equilibrium system: Onsager relation $R_{AB} = R_{BA}$).

• (1), (3) $\Rightarrow$ $R_{AA} > D_A q^2$.

$\Rightarrow$ $\boxed{\text{local activation}}$

Can we find sufficient cond.?

$R_{AA} > 0$.

$R_{BB} < 0$.

$|R_{AB} R_{BA}| > R_{AA} |R_{BB}|$

$R_{AA} > D_A q^2$.

$D_B >> D_A$.

(3).

local activation

global inhibition.