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Does dynamics reflect topology in directed networks?

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Abstract. – We present and analyze a topologically induced transition from ordered, synchronized to disordered dynamics in directed networks of oscillators. The analysis reveals where in the space of networks this transition occurs and its underlying mechanisms. If disordered, the dynamics of the units is precisely determined by the topology of the network and thus characteristic for it. We develop a method to predict the disordered dynamics from topology. The results suggest a new route towards understanding how the precise dynamics of the units of a directed network may encode information about its topology.

Networks of interacting units prevail in a variety of systems, ranging from gene regulatory networks and neural networks to food webs and the world wide web [1, 2]. A fundamental question is: What kind of dynamics can we expect given a network of prescribed connection topology [3]? Even in networks of known dynamical units, known type of interactions between them and known topological details, it is hard to infer which kind of typical collective dynamics the network will display (cf. refs. [3–6]). If parts of the network exhibit residual symmetries, such as permutation or translation invariance, some general properties of the dynamics can be deduced [6]. If, however, no symmetries remain, it is still an open question how topological factors can control network dynamics. See, *e.g.*, [7–9] for some interesting recent approaches for phase oscillator networks of specific connectivities.

In this letter, we study directed networks of phase oscillators that exhibit a mechanism to synchronize and reveal general principles about how topology controls dynamics: Specifically, in networks with an invariant state of in-phase synchrony we analyze how the topology can control the units' dynamics in a neighborhood of synchrony. We find that such networks, depending on their coarse-scale topological properties, belong to one of two classes exhibiting very different long-term dynamics: Networks of class I show in-phase synchrony in which each unit displays identical dynamics, independent of the unit's *topological identity* [10], *i.e.* independent of where in the network it is located. Networks of class II, instead of synchrony, show disordered dynamics. Here, together with the initial condition, the fine-scale topology precisely controls the dynamics of each unit. The dynamics therefore strongly depends on where the unit is located in the network —its topological identity. We develop a method to predict the disordered dynamics from the network's topology. Due to their topological

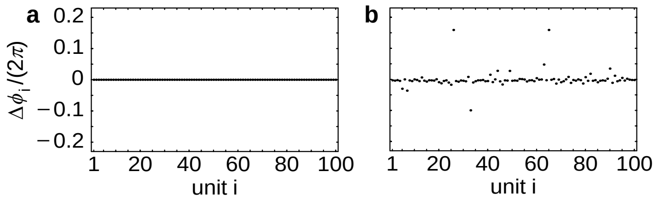

Fig. 1 – Synchronization-disorder transition in directed network dynamics. (a), (b) The long-time dynamics of two different random network realizations of $N = 100$ units with the same connection probability $p = 0.05$ started from the same random initial condition. The phase differences $\Delta\phi_i := \phi_i - \langle\phi_j\rangle_j$ with respect to the average phase $\langle\phi_j\rangle_j$ of all units are shown *vs.* the units i, plotted relative to their possible range, 2π.

origin, both the separation of the ensemble of networks into two unique classes and the specific disordered dynamics realized by a network appear to be general phenomena and not restricted to the system studied here.

Let us elaborate these findings. Consider a network of N phase oscillators i that interact via directed connections. The network topology is arbitrary and determined by the sets $\mathrm{In}(i)$ of those units j that have input connections to i, denoted $j \to i$. We analyze a simple, paradigmatic model of interacting periodic oscillators, the Kuramoto model [11–13] defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_i(t) = \omega_i + \sum_{j\in\mathrm{In}(i)} J_{ij}\sin(\phi_j - \phi_i), \tag{1}$$

where the phase variable $\phi_i(t) \in [0, 2\pi)$ (with periodic boundary conditions) determines the state of unit i at time t, ω_i is its frequency, and $J_{ij} \geq 0$ is the strength of coupling from j to i with $J_{ij} = 0$ if there is no connection $j \to i$. In order to stress the topological effects, we neglect inhomogeneities in the dynamical parameters: we consider identical units $\omega_i = \omega$ and homogeneous total input coupling strengths such that $\sum_j J_{ij} = J$ (in all illustrating examples we choose $J_{ij} = J/k_i$ if unit i receives k_i input connections from other units j). Without loss of generality, we take $J = 1$ in the following. We consider initial states in a neighborhood of the in-phase synchronous solution to reveal those features that apply for other oscillator networks as well.

Observing the dynamics (1) on different topologies, it was intriguing to find that seemingly similar networks (such as realizations of networks with identical degree distribution) yet displayed very different dynamics. Consider, for instance, the long-term dynamics of random networks in which every connection $j \to i$ is present with probability p. Several such networks show in-phase synchrony (cf. fig. 1a). Thus, the final states of the units display no information about the network topology. The units' *topological identity* is hidden. Other networks with identical statistical properties display disordered periodic dynamics, even when initialized in the same state (fig. 1b). In such disordered states almost every unit displays a different phase. It turns out (see below) that the network topology precisely controls the dynamics of these units. The units' dynamics thus display their topological identity!

This phenomenon raises a number of questions. In which networks and how does the disordered state emerge? What determines the individual units' dynamics if the network displays disorder?

To answer these questions, we studied the dynamics of small networks that exhibit qualitatively the same disordered state (see, *e.g.*, fig. 2a). We tried to find a systematic dependence of the units' states on the network topology. As a first step, we ordered the units of the network (fig. 2a) such that units with similar phases are displayed at proximate positions

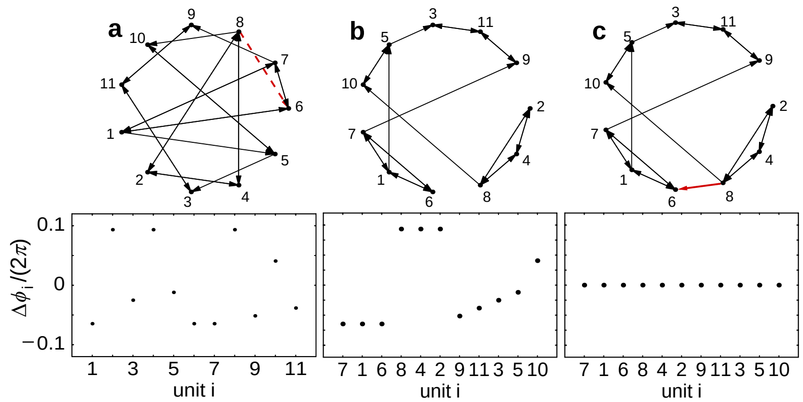

Fig. 2 – (Color online) Dynamics of small networks ($N = 11$) started from the same random initial condition. The upper column of each panel displays the directed networks with units labeled $i \in \{1, \dots, N\}$. The lower displays the relative phase differences $\Delta\phi_i/(2\pi)$ *vs.* i. (a) A network with homogeneous in-degree $k_i = 2$ for all units i exhibits an irregular asymptotic state. (b) Same network as in (a) with units with similar phases grouped. (c) A network with only one more directed edge ($8 \to 6$) (red; red dashed edge in (a)) compared to that of (b) induces a completely ordered state with identical dynamics of all units.

(fig. 2b). This ordering reveals a division of the network in terms of its strongly connected components (SCCs) [14]. Whereas there are some units with identical phases, several phases appear uniquely. In a network with just one more edge (fig. 2c), the collective dynamics is completely synchronized such that the topological identity of all units is hidden (cf. also fig. 1a). The above finding that ordering of the phases seems to reveal information about the coarse scale network topology led us to hypothesize that the partition of a network into SCCs is important to understand its dynamics.

To test this hypothesis, we first analyze the dynamics of networks of arbitrary connectivity in a neighborhood of in-phase synchrony ($\phi_i(t) = \phi_0(t)$ for all units i and all times t). Disconnected parts of a network can be treated independently, such that we focus on connected networks here. Sufficiently small perturbations $\delta_i(t) := \phi_i(t) - \phi_0(t)$ to the synchronous state satisfy

$$\dot{\delta}_i = \sum_{j \in \mathrm{In}(i)} J_{ij} \sin(\delta_j - \delta_i) \tag{2}$$

for all i, which in first-order approximation reads $\dot{\delta}_i = -J\delta_i + \sum_{j\in\mathrm{In}(i)} J_{ij}\delta_j$, or $\dot{\boldsymbol{\delta}} \doteq M\boldsymbol{\delta}$ in matrix form, where

$$M_{ij} = \begin{cases} -J & \text{if } j = i, \\ J_{ij} & \text{if } j \in \mathrm{In}(i), \\ 0 & \text{if } j \notin \{i\} \cup \mathrm{In}(i) \end{cases} \tag{3}$$

are the matrix elements of M and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_N)^{\mathsf{T}}$ is the vector of the individual units' perturbations δ_i. This results in the first-order period-T map ($T = 2\pi/\omega$) given by

$$\boldsymbol{\delta}(T) = A\boldsymbol{\delta}(0), \tag{4}$$

where the matrix elements of $A = e^{MT}$ satisfy $A_{ij} \geq 0$, reflecting the attractive couplings $J_{ij} \geq 0$, and $\sum_j A_{ij} = 1$ due to time translation invariance of the periodic orbit.

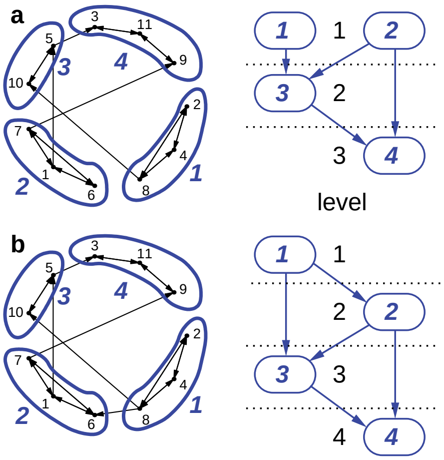

Fig. 3

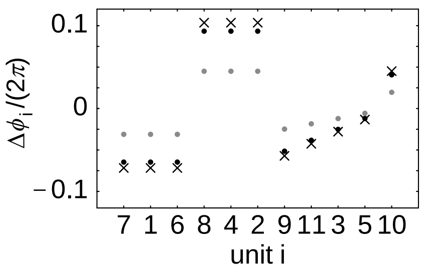

Fig. 4

Fig. 3 – (Color online) Decomposition of networks shown in fig. 2 in terms of their strongly connected components (SCCs). On the left, the vertices of the networks are grouped to SCCs $s \in \{1,2,3,4\}$ (large italic numbers). On the right, the level structure of these components is shown. (a) Three-level network with two source, cf. figs. 2a, b. (b) One additional link makes it a four-level network with one source, cf. fig. 2c.

Fig. 4 – Prediction of the dynamics in a disordered state (cf. fig. 2a, b) based on the composition analysis. The relative phase difference $\Delta\phi_i/(2\pi)$ is shown for the grouped units i. The linear prediction ($\times$) of the actual phases ($\bullet$) (based on one intial state) well distinguishes the ordered from the disordered state (which would be a constant at zero) and even is a good indicator of the quantitative dynamics of the units. The asymptotic phase dynamics started from a different initial state (gray circles) illustrates that in this example other initial states yield a pattern that is distinguished from the former pattern only by a real mulitplicative factor (in first-order approximation).

For networks of arbitrary connectivities, this implies, via the Geršgorin disk theorem [15], that all eigenvalues λ_i of A satisfy $|\lambda_i| \leq 1$. A sufficiently small perturbation to the synchronous state cannot grow (in maximum norm), cf. [5], such that synchrony is at least marginally stable. Moreover, independent of the network connectivity there is one eigenvalue $\lambda_1 = 1$ with an eigenvector $\boldsymbol{v}_1 = (1,1,\ldots,1)$ corresponding to the uniform phase shift.

If the network is strongly connected [14] the Perron-Frobenius theorem [15] guarantees that the largest eigenvalue $\lambda_1 = 1$ is unique and all other eigenvalues satisfy $|\lambda_i| < 1$ for $i \in \{2,\ldots,N\}$. This implies that the synchronized state is asymptotically stable and thus locally attracting. In networks of irregular topology we even often find that the system converges towards it from arbitrary initial conditions.

If the network is not strongly connected it consists of two or more strongly connected components (SCCs) and the analysis of the asymptotic dynamics is more involved. For better accessibility of the main points of this letter we describe the details of this analysis in the appendix. Briefly, for a given network, we first determine the SCCs and the uni-directional connections among them. Second, we determine the level structure of this super-network of SCCs (cf. fig. 3). Based on this composition analysis we have revealed a number of distinctive features of the dynamics on directed networks.

The ensemble of networks divides into two classes with qualitatively different-long-term dynamics (initialized sufficiently close to the in-phase solution). All networks that have $M = 1$ source SCC (which does not receive any input from other SCCs) belong to class I: this source

SCC is guaranteed to synchronize because it itself is a strongly connected network without further input. Since each unit i performs a local weighted averaging of phases determined by the weights A_{ij} in (4), all units outside the source component asymptotically converge towards the (common) phase of the units within the (only) source SCC. This result also follows explicitly from the analysis given in the appendix in the special case of only one source component in level $\ell = 1$ (and no source components in levels $\ell > 1$). It implies that for all networks with one source SCC the local asymptotic dynamics is also in-phase synchrony.

In contrast, networks having $M \geq 2$ source SCCs (class II) typically show disordered dynamics. These M source SCCs can synchronize independently of each other, creating $M-1$ independent phase differences which result in an $(M-1)$-dimensional continuous family of periodic orbits, that include the synchronous state as only *one* specific orbit. All these orbits are marginally stable, in particular the synchronous state has a basin of attraction of measure zero, such that the dynamics is almost surely disordered. For the examples above, we find that the dynamics shown in fig. 1a originates from a class-I network whereas that of fig. 1b originates from a class-II network.

The composition analysis also reveals how the details of the topology of the network precisely control its dynamics in the disordered state: The topological identity of each unit, particularly the fine-scale topology of that SCC it is part of, determines the unit's dynamics. In fact, we can predict the disordered dynamics on a fine scale: Given the initial state $\boldsymbol{\phi}(0)$, we uniquely determine the approximate phases of all units recursively level by level, and hence predict the complete collective dynamics of the network from the topological identity of their units (see the appendix). Figure 4 illustrates such a prediction. It resembles well the actual dynamics of the units.

Reversely, partial information about the topology of the network may be obtained from knowing the disordered dynamics of its units: Iterating eq. (5), we obtain explicit linear restrictions of the space of all networks from the disordered dynamics by imposing its invariance. So only a lower-dimensional subset of networks is consistent with the phase pattern.

What is the mechanism underlying the transition to topology-induced disorder? The following description is general; nevertheless it is instructive to imagine, as an illustrating example, a network composed of two source SCCs and one sink SCC which receives input from the other two. A strongly connected network, and thus each source SCC, synchronizes completely. However, different of these source SCCs typically converge towards different phases, that depend on the initial state. If now different units in a downstream SCC are pulled towards different phases, and there is a complicated pattern of connections between them within this SCC, the dynamics of all its units will typically be distinct. In particular, the units' dynamics depend on the phases of the units in connected upstream SCCs, *i.e.* indirectly on the initial state of the network and on the specific topology of the SCC considered.

All these phenomena appear to be general and are not restricted to the model system (1) considered here. This is due to the topological origin of the phenomena: First, the transition line between networks of classes I and II is identical for various kinds of oscillator networks having an invariant in-phase solution. Second, given an initial state sufficiently close to synchrony, the disorder in the long-term dynamics is *characteristic for the topology* of a network: The linear analysis (see the appendix) holds as well for all disordered dynamics that are topologically equivalent within the class of periodic single-variable oscillator networks. We checked this explicitly for networks of i) Kuramoto oscillators (1) with coupling functions different from the sine function and ii) spiking neural oscillators where interactions are delayed and mediated by pulses that occur only at discrete instances of time [5,16]. Although we have no proof of how general these results are beyond single-variable oscillators, we also observed that even iii) networks of diffusively coupled chaotic Rössler systems [17] behave similarly. On the same network topology, the dynamics of these three kinds of distinct systems show closely related patterns of phase disorder.

Commonly, transitions from synchrony to disordered dynamics have been devoted to heterogeneities of, *e.g.*, dynamical parameters or the degree distribution, cf. [18–20]. However, the precise impact of topology onto the dynamics of directed networks revealed here was so far not noticed. Even recent studies, considering the exact dynamics of networks of given, specific topologies (see, *e.g.* [5,21] and references therein) have not taken notice of this impact. The main reason for this may be that all example networks chosen to look at explicitly again were standard cases such as highly connected random networks or lattices.

Real-world oscillator networks, occurring across disciplines in physics, biology and technology [3, 13], however, have a far more complicated topology, and, as demonstrated in this letter, may thus strongly deviate in their dynamics. In a study [22] related to ours, the notion of long-range action has been introduced showing that in certain directed networks of iterated maps the dynamics of boundary units can control the dynamics of the entire network. Our results suggest that local and global topological features, such as the SCC super-network and the detailed topology of particular SCCs, may act together to precisely control the dynamics of individual units in complex directed networks. The concepts developed here may thus also help to uncover information about the topologies of such networks from their dynamics.

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Appendix

Level structure of the SCC super-network. – The level structure of the SCC super-network is constructed in three steps. First, we determine the SCCs of the network using a standard method [23], the computational complexity of which is $\mathcal{O}(N)$. Second, connections between them are straightforwardly derived from the underlying connections between units comprising these SCCs. A connection from one SCC to another, $s \to s'$ is present if there are $i \in s$ and $j \in s'$ with a connection, $i \to j$ between them. Third, we find the longest undirected path from any source SCC (without incoming connections) to any sink SCC (without outgoing connections). The length of such a path is found by counting a connection followed along its direction as "+1" and against its direction as "−1". All units i in a source SCC of the longest path is given the level number $\ell(i) = 1$. The levels of all other SCCs are determined recursively according to the above counting rule. The computational costs of finding the inter-SCC connections and the level structure strongly depend on the network under consideration.

Dynamics from network topology. – Given the level structure, the linearized dynamics of every unit is determined for all units in every given level, starting with level $\ell = 1$ and proceeding through subsequent levels recursively. Let $\boldsymbol{\phi} = \left(\boldsymbol{\phi}^{(1)}, \ldots, \boldsymbol{\phi}^{(L)}\right) = (\phi_{i_1}, \ldots, \phi_{i_N})$ denote the asymptotic phases of all units ϕ_i in terms of the collection of phases $\boldsymbol{\phi}^{(\ell)}$ of the units at a given level $\ell \in \{1, \ldots, L\}$. For all units i with $\ell(i) = 1$, their final states are $\phi_i^{(1)} = c_s$, where c_s depends on the initial state $\boldsymbol{\phi}_s(0)$ restricted to the SCC s. It equals the first component of the vector $\boldsymbol{c}_s = V^{-1}\boldsymbol{\phi}_s(0)$, where $V = (\boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_R})$ is a matrix of the R

eigenvectors $\boldsymbol{v}_{i_r}$ of A localized on the SCC s and $\boldsymbol{v}_{i_1}$ is the eigenvector corresponding to the eigenvalue $\lambda_{i_1} = 1$. This yields the vector $\boldsymbol{\phi}^{(1)}$ of asymptotic phases in all units in level $\ell = 1$.

The phases $\boldsymbol{\phi}^{(\ell)}$ of units in the other levels $\ell \geq 2$ are determined iteratively given the phases $\boldsymbol{\phi}^{(\ell-1)}$ of units in level $\ell - 1$. If some of the $\boldsymbol{\phi}^{(\ell)}$ are part of a source SCC in level ℓ, these phases $\boldsymbol{\phi}^{(\ell)}_{\text{source}}$ are determined analogous to those in level $\ell = 1$. The corresponding sub-matrices $A_{\ell,\ell-1}$ and $A_{\ell,\ell}$ of the matrix A in (4) needed to determine the remaining phases $\boldsymbol{\phi}^{(\ell)}_{\text{no source}}$ describe the interactions with units of the previous level $\ell - 1$, and within the SCCs of the current level ℓ, respectively. Note that by definition of the level structure, there are no interactions from level ℓ to level $\ell - 1$. Thus the equation encoding this uni-directional dependence, $\boldsymbol{\phi}^{(\ell)}_{\text{no source}} = A_{\ell,\ell-1}\boldsymbol{\phi}^{(\ell-1)} + A_{\ell,\ell}\boldsymbol{\phi}^{(\ell)}_{\text{no source}}$, yields

$$\boldsymbol{\phi}^{(\ell)}_{\text{no source}} = (1 - A_{\ell,\ell})^{-1} A_{\ell,\ell-1} \boldsymbol{\phi}^{(\ell-1)} \tag{5}$$

such that, together with the $\boldsymbol{\phi}^{(\ell)}_{\text{source}}$ from above, all phases $\boldsymbol{\phi}^{(\ell)}$ of units in level ℓ are determined. Iterating this for all levels $\ell \in \{2, \ldots, L\}$ we obtain the linear prediction of the complete disordered asymptotic state $\boldsymbol{\phi}$. This analysis only depends on the linearized effective couplings A_{ij} that determine the SCC super-network and is thus not restricted to the system (1).

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